

## Chapter 6

# The Chaotic Motion of Dynamical Systems

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We study simple nonlinear deterministic models that exhibit chaotic behavior. We will find that the use of the computer to do numerical experiments will help us gain insight into the nature of chaos.

### 6.1 Introduction

Most natural phenomena are intrinsically nonlinear. Weather patterns and the turbulent motion of fluids are everyday examples. Although we have explored some of the properties of nonlinear systems in Chapter 4, it is easier to introduce some of the important concepts in the context of ecology. Our first goal will be to motivate and analyze the one-dimensional difference equation

$$x_{n+1} = 4rx_n(1 - x_n), \quad (6.1)$$

where  $x_n$  is the ratio of the population in the  $n$ th generation to a reference population. We shall see that the dynamical properties of (6.1) are surprisingly intricate and have important implications for the development of a more general description of nonlinear phenomena. The significance of the behavior of (6.1) is indicated by the following quote from the ecologist Robert May:

*“Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student’s intuition about nonlinear systems. Not only in research but also in the everyday world of politics and economics we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.”*

The study of chaos is of much current interest, but the phenomena is not new and has been of interest, particularly to astronomers and mathematicians, for over one hundred years. Much of

the current interest is due to the use of the computer as a tool for making empirical observations. We will use the computer in this spirit.

## 6.2 A Simple One-Dimensional Map

Many biological populations effectively consist of a single generation with no overlap between successive generations. Imagine an island with an insect population that breeds in the summer and leaves eggs that hatch the following spring. Because the population growth occurs at discrete times, it is appropriate to model the population growth by a difference equation rather than by a differential equation. A simple model of population growth that relates the population in generation  $n + 1$  to the population in generation  $n$  is given by

$$P_{n+1} = aP_n, \quad (6.2)$$

where  $P_n$  is the population in generation  $n$  and  $a$  is a constant. In the following, we assume that the time interval between generations is unity, and will refer to  $n$  as the time.

If  $a > 1$ , each generation will be  $a$  times larger than the previous one. In this case (6.2) leads to geometrical growth and an unbounded population. Although the unbounded nature of geometrical growth is clear, it is remarkable that most of us do not integrate our understanding of geometrical growth into our everyday lives. Can a bank pay 4% interest each year indefinitely? Can the world's human population grow at a constant rate forever?

It is natural to formulate a more realistic model in which the population is bounded by the finite carrying capacity of its environment. A simple model of density-dependent growth is

$$P_{n+1} = P_n(a - bP_n). \quad (6.3)$$

Equation (6.3) is nonlinear due to the presence of the quadratic term in  $P_n$ . The linear term represents the natural growth of the population; the quadratic term represents a reduction of this natural growth caused, for example, by overcrowding or by the spread of disease.

It is convenient to rescale the population by letting  $P_n = (a/b)x_n$  and rewriting (6.3) as

$$x_{n+1} = ax_n(1 - x_n). \quad (6.4)$$

The replacement of  $P_n$  by  $x_n$  changes the units used to define the various parameters. To write (6.4) in the form (6.1), we define the parameter  $r = a/4$  and obtain

$$x_{n+1} = f(x_n) = 4rx_n(1 - x_n). \quad (6.5)$$

The rescaled form (6.5) has the desirable feature that its dynamics are determined by a single control parameter  $r$ . Note that if  $x_n > 1$ ,  $x_{n+1}$  will be negative. To avoid this unphysical feature, we impose the conditions that  $x$  is restricted to the interval  $0 \leq x \leq 1$  and  $0 < r \leq 1$ . Because the function  $f(x)$  defined in (6.5) transforms any point on the one-dimensional interval  $[0, 1]$  into another point in the same interval, the function  $f$  is called a *one-dimensional map*. The form of  $f(x)$  in (6.5) is known as the *logistic map*. The logistic map is a simple example of a *dynamical system*, that is, the map is a deterministic, mathematical prescription for finding the future state of a system given its present state.

The sequence of values  $x_0, x_1, x_2, \dots$  is called the *trajectory*. To check your understanding, suppose that the initial value of  $x_0$  or *seed* is  $x_0 = 0.5$  and  $r = 0.2$ . Do a calculation to show that the trajectory is  $x_1 = 0.2, x_2 = 0.128, x_3 = 0.089293, \dots$ . The first thirty iterations of (6.5) are shown for two values of  $r$  in Figure 6.1.

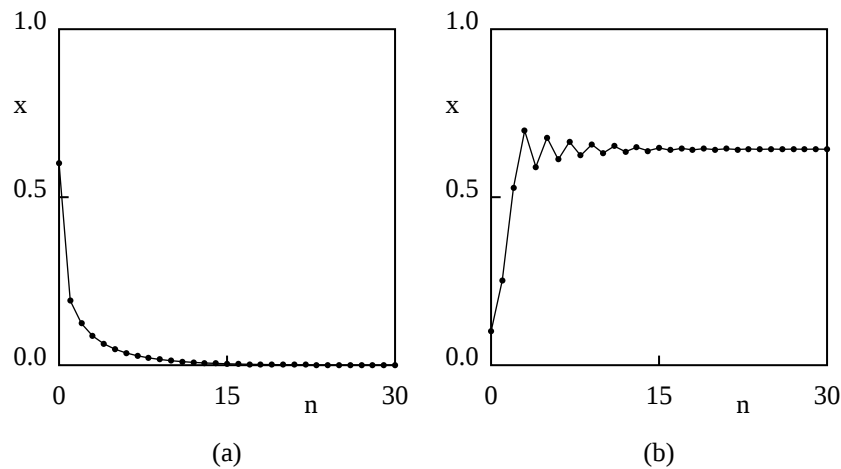


Figure 6.1: (a) The trajectory of  $x$  for  $r = 0.2$  and  $x_0 = 0.6$ . Note that the stable fixed point is  $x = 0$ . (b) The trajectory for  $r = 0.7$  and  $x_0 = 0.1$ . Note the initial transient behavior. The lines between the points are a guide to the eye.

The class `IterateMapApp` computes the trajectory of the logistic map in (6.5). Note that we have extended the `AbstractCalculation` class, which is appropriate because many of the results of Sections 6.1–6.4 were discovered using a programmable calculator.

Listing 6.1: The `IterateMapApp` program iterates the logistic map and plots the resulting trajectory.

```
package org.opensourcephysics.sip.ch06;
import org.opensourcephysics.controls.AbstractCalculation;
import org.opensourcephysics.controls.CalculationControl;
import org.opensourcephysics.frames.PlotFrame;

public class IterateMapApp extends AbstractCalculation {
    int datasetIndex = -1;
    PlotFrame plotFrame = new PlotFrame("iterations", "x", "trajectory");
    public IterateMapApp() {
        plotFrame.setMarkerSize(1, 1);
    }

    public void reset() {
        plotFrame.clearData();
        plotFrame.repaint();
        control.setValue("r", 0.2);
    }
}
```

```

        control.setValue("x", 0.6);
        control.setValue("iterations", 50);
    }

    public void calculate() {
        double r = control.getDouble("r");
        double x = control.getDouble("x");
        int iterations = control.getInt("iterations");
        datasetIndex++;
        for(int i = 0; i <= iterations; i++) {
            plotFrame.append(datasetIndex, i, x);
            x = map(r, x);
        }
    }

    double map(double r, double x) {
        return 4*r*x*(1-x); // iterate map
    }

    public static void main(String[] args) {
        CalculationControl.createApp(new IterateMapApp());
    }
}

```

In Problems 6.1 and 6.3 we use `IterateMapApp` to explore the dynamical properties of the logistic map.

**Problem 6.1.** Exploration of period-doubling

- a. Explore the dynamical behavior of (6.5) with  $r = 0.24$  for different values of  $x_0$ . Show that  $x = 0$  is a *stable fixed point*. That is, the iterated values of  $x$  converge to  $x = 0$  independently of the value of  $x_0$ . If  $x$  represents the population of insects, describe the qualitative behavior of the population.
- b. Explore the dynamical behavior of (6.5) for  $r = 0.26, 0.5, 0.74$ , and  $0.748$ . A fixed point is *unstable* if for almost all values of  $x_0$  near the fixed point, the trajectories diverge from it. Verify that  $x = 0$  is an unstable fixed point for  $r > 0.25$ . Show that for the suggested values of  $r$ , the iterated values of  $x$  do not change after an initial *transient*, that is, the long time dynamical behavior is *period 1*. In Appendix 6A we show that for  $r < 3/4$  and for  $x_0$  in the interval  $0 < x_0 < 1$ , the trajectories approach the *stable attractor* at  $x = 1 - 1/4r$ . The set of initial points that iterate to the attractor is called the *basin* of the attractor. For the logistic map, the interval  $0 < x < 1$  is the basin of attraction of the attractor  $x = 1 - 1/4r$ .
- c. Explore the dynamical properties of (6.5) for  $r = 0.752, 0.76, 0.8$ , and  $0.862$ . For  $r = 0.752$  and  $0.862$  approximately 1000 iterations are necessary to obtain convergent results. Show that if  $r$  is increased slightly beyond  $0.75$ ,  $x$  oscillates between two values after an initial transient behavior. That is, instead of a stable cycle of period 1 corresponding to one fixed point, the system has a stable cycle of period 2. The value of  $r$  at which the single fixed point  $x^*$  splits or

bifurcates into two values  $x_1^*$  and  $x_2^*$  is  $r = b_1 = 3/4$ . The pair of  $x$  values,  $x_1^*$  and  $x_2^*$ , form a *stable attractor* of period 2.

- d. What are the stable attractors of (6.5) for  $r = 0.863$  and  $0.88$ ? What is the corresponding period?
- e. What are the stable attractors and corresponding periods for  $r = 0.89$ ,  $0.891$ , and  $0.8922$ ?

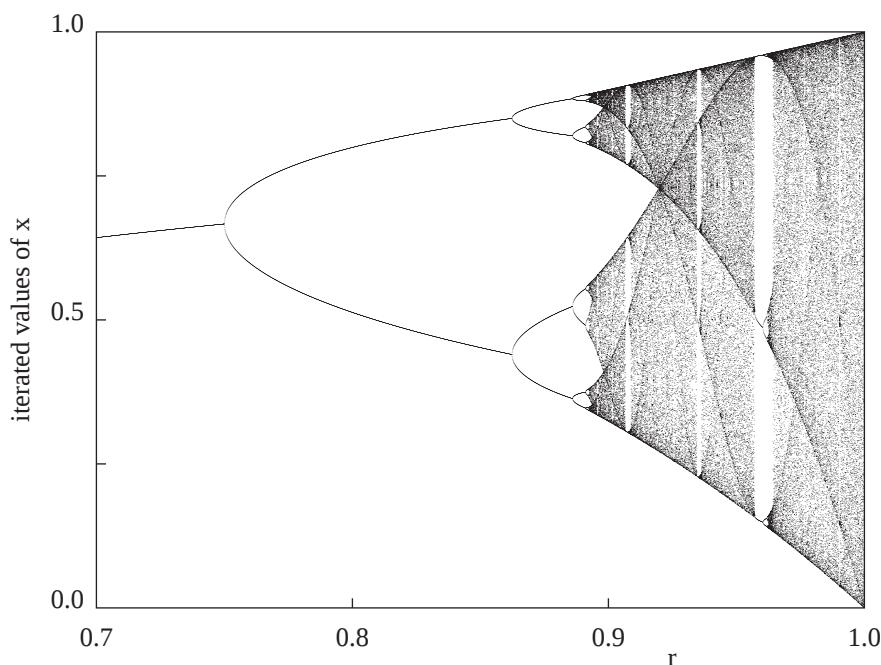


Figure 6.2: Bifurcation diagram of the logistic map. For each value of  $r$ , the iterated values of  $x_n$  are plotted after the first 1000 iterations are discarded. Note the transition from periodic to chaotic behavior and the narrow windows of periodic behavior within the region of chaos.

Another way to determine the behavior of (6.5) is to plot the values of  $x$  as a function of  $r$  (see Figure 6.2). The iterated values of  $x$  are plotted after the initial transient behavior is discarded. Such a plot is generated by `BifurcateApp`. For each value of  $r$ , the first `ntransient` values of  $x$  are computed but not plotted. Then the next `nplot` values of  $x$  are plotted, with the first half in one color and the second half in another. Why? This process is repeated for a new value of  $r$  until the desired range of  $r$  values is reached. A typical value of `ntransient` should be in the range of 100–1000 iterations. The magnitude of `nplot` should be at least as large as the longest period that you wish to observe. `BifurcateApp` extends `AbstractSimulation` rather than `AbstractCalculation` because the calculations can become time consuming, and hence you might want to stop them before they are finished and reset some of the parameters.

Listing 6.2: The `BifurcateApp` program generates a bifurcation plot of the logistic map.

```

package org.opensourcephysics.sip.ch06;
import org.opensourcephysics.controls.AbstractSimulation;
import org.opensourcephysics.controls.SimulationControl;
import org.opensourcephysics.frames.PlotFrame;

public class BifurcateApp extends AbstractSimulation {
    double r;        // control parameter
    double dr;        // incremental change of r, suggest dr <= 0.01
    int ntransient;   // number of iterations not plotted
    int nplot;        // number of iterations plotted
    PlotFrame plotFrame = new PlotFrame("r", "x", "Bifurcation diagram");
    public BifurcateApp() {
        plotFrame.setMarkerSize(0, 0); // small size gives better resolution
        plotFrame.setMarkerSize(1, 0);
    }

    public void initialize () {
        plotFrame.clearData();
        r = control.getDouble("initial r");
        dr = control.getDouble("dr");
        ntransient = control.getInt("nttransient");
        nplot = control.getInt("nplot");
    }

    public void doStep() {
        if(r<1.0) {
            double x = 0.5;
            for(int i = 0; i<ntransient; i++) { // x values not plotted
                x = map(x, r);
            }
            for(int i = 0; i<nplot/2; i++) { // plot half the points in dataset zero
                x = map(x, r);
                plotFrame.append(0, r, x); // show different x-values for given value of r
            }
            for(int i = nplot/2+1; i<nplot; i++) { // plot remaining points in dataset one
                x = map(x, r);
                plotFrame.append(1, r, x); // dataset one has a new color
                i++;
            }
            r += dr;
        }
    }

    public void reset() {
        control.setValue("initial r", 0.2);
        control.setValue("dr", 0.005);
        control.setValue("nttransient", 200);
        control.setValue("nplot", 50);
    }
}

```

```

    }

    double map(double x, double r) {
        return 4*r*x*(1-x);
    }

    public static void main(String[] args) {
        SimulationControl.createApp(new BifurcateApp());
    }
}

```

**Problem 6.2.** Qualitative features of the logistic map

- Use `BifurcateApp` to identify period 2, period 4, and period 8 behavior as can be seen in Figure 6.2. It might be necessary to “zoom in” on a portion of the plot. How many period-doublings can you find?
- Change the scale so that you can follow the iterations of  $x$  from period 4 to period 16 behavior. How does the plot look on this scale in comparison to the original scale?
- Describe the shape of the trajectory near the bifurcations from period 2  $\rightarrow$  period 4, period 4  $\rightarrow$  8, etc. These bifurcations are frequently called *pitchfork bifurcations*.

The bifurcation diagram in Figure 6.2 indicates that the period-doubling behavior ends at  $r \approx 0.892$ . This value of  $r$  is known very precisely and is given by  $r = r_\infty = 0.892486417967\dots$ . At  $r = r_\infty$ , the sequence of period-doublings accumulate to a trajectory of infinite period. In Problem 6.3 we explore the behavior of the trajectories for  $r > r_\infty$ .

**Problem 6.3.** Chaotic behavior

- For  $r > r_\infty$ , two initial conditions that are very close to one another can yield very different trajectories after a small number of iterations. As an example, choose  $r = 0.91$  and consider  $x_0 = 0.5$  and  $0.5001$ . How many iterations are necessary for the iterated values of  $x$  to differ by more than ten percent? What happens for  $r = 0.88$  for the same choice of seeds?
- The accuracy of floating point numbers retained on a digital computer is finite. To test the effect of the finite accuracy of your computer, choose  $r = 0.91$  and  $x_0 = 0.5$  and compute the trajectory for 200 iterations. Then modify your program so that after each iteration, the operation  $\mathbf{x} = \mathbf{x}/10$  is followed by  $\mathbf{x} = 10*\mathbf{x}$ . This combination of operations truncates the last digit that your computer retains. Compute the trajectory again and compare your results. Do you find the same discrepancy for  $r < r_\infty$ ?
- What are the dynamical properties for  $r = 0.958$ ? Can you find other windows of periodic behavior in the interval  $r_\infty < r < 1$ ?

### 6.3 Period-Doubling

The results of the numerical experiments that we did in Section 6.2 probably have convinced you that the dynamical properties of a simple nonlinear deterministic system can be quite complicated.

To gain more insight into how the dynamical behavior depends on  $r$ , we introduce a simple graphical method for iterating (6.5). In Figure 6.3 we show a graph of  $f(x)$  versus  $x$  for  $r = 0.7$ . A diagonal line corresponding to  $y = x$  intersects the curve  $y = f(x)$  at the two fixed points  $x^* = 0$  and  $x^* = 9/14 \approx 0.642857$  (see (6.6b)). If  $x_0$  is not fixed point, we can find the trajectory in the following way. Draw a vertical line from  $(x = x_0, y = 0)$  to the intersection with the curve  $y = f(x)$  at  $(x_0, y_0 = f(x_0))$ . Next draw a horizontal line from  $(x_0, y_0)$  to the intersection with the diagonal line at  $(y_0, y_0)$ . On this diagonal line  $y = x$ , and hence the value of  $x$  at this intersection is the first iteration  $x_1 = y_0$ . The second iteration  $x_2$  can be found in the same way. From the point  $(x_1, y_0)$ , draw a vertical line to the intersection with the curve  $y = f(x)$ . Keep  $y$  fixed at  $y = y_1 = f(x_1)$ , and draw a horizontal line until it intersects the diagonal line; the value of  $x$  at this intersection is  $x_2$ . Further iterations can be found by repeating this process.

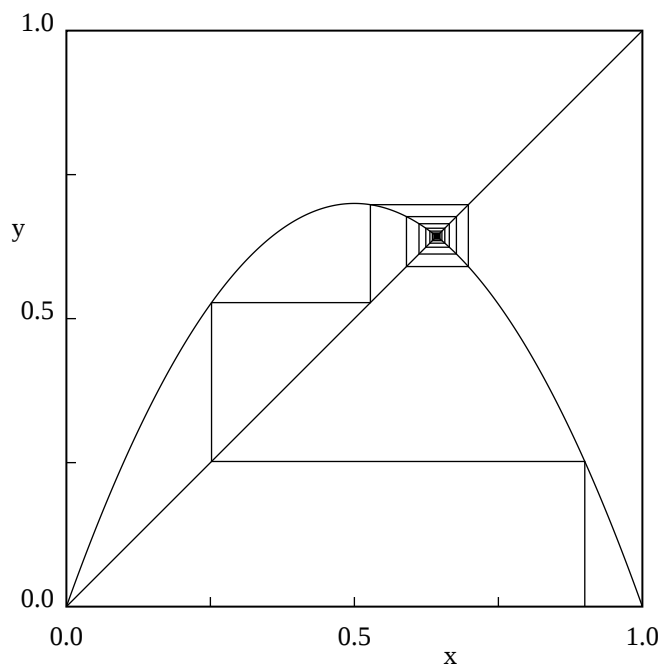


Figure 6.3: Graphical representation of the iteration of the logistic map (6.5) with  $r = 0.7$  and  $x_0 = 0.9$ . Note that the graphical solution converges to the fixed point  $x^* \approx 0.643$ .

This graphical method is illustrated in Figure 6.3 for  $r = 0.7$  and  $x_0 = 0.9$ . If we begin with any  $x_0$  (except  $x_0 = 0$  and  $x_0 = 1$ ), the iterations will converge to the fixed point  $x^* \approx 0.643$ . Repeat the procedure shown in Figure 6.3 by hand and convince yourself that you understand the nature of the graphical solution of the iterated values of the map. For this value of  $r$ , the fixed

point is stable (an attractor of period 1). In contrast, no matter how close  $x_0$  is to the fixed point at  $x = 0$ , the iterates diverge away from it, and this fixed point is unstable.

How can we explain the qualitative difference between the fixed point at  $x = 0$  and at  $x^* = 0.642857$  for  $r = 0.7$ ? We know that the local slope of the curve  $y = f(x)$  determines the distance moved horizontally each time  $f$  is iterated. A slope steeper than  $45^\circ$  leads to a value of  $x$  further away from its initial value. Hence, the criterion for the stability of a fixed point is that the magnitude of the slope at the fixed point must be less than  $45^\circ$ . That is, if  $|df(x)/dx|_{x=x^*}$  is less than unity, then  $x^*$  is stable; conversely, if  $|df(x)/dx|_{x=x^*}$  is greater than unity, then  $x^*$  is unstable. An inspection of  $f(x)$  in Figure 6.3 shows that  $x = 0$  is unstable because the slope of  $f(x)$  at  $x = 0$  is greater than unity. In contrast, the magnitude of the slope of  $f(x)$  at  $x = x^* \approx 0.643$  is less than unity and this fixed point is stable. In Appendix 6A, we use similar analytical arguments to show that

$$x^* = 0 \text{ is stable for } 0 < r < 1/4, \quad (6.6a)$$

and

$$x^* = 1 - \frac{1}{4r} \text{ is stable for } 1/4 < r < 3/4. \quad (6.6b)$$

Thus for  $0 < r < 3/4$ , the eventual behavior after many iterations is known.

What happens if  $r$  is greater than  $3/4$ ? We found in Section 6.2 that if  $r$  is slightly greater than  $3/4$ , the fixed point of  $f$  becomes unstable and gives birth (bifurcates) to a cycle of period 2. Now  $x$  returns to the same value only after every second iteration, and the fixed points of  $f(f(x))$  are the stable attractors of  $f(x)$ . In the following, we write  $f^{(2)}(x) = f(f(x))$  and  $f^{(n)}(x)$  for the  $n$ th iterate of  $f(x)$ . (Do not confuse  $f^{(n)}(x)$  with the  $n$ th derivative of  $f(x)$ .) For example, the second iterate  $f^{(2)}(x)$  is given by the fourth-order polynomial:

$$\begin{aligned} f^{(2)}(x) &= 4r[4rx(1-x)] - 4r[4rx(1-x)]^2 \\ &= 4r[4rx(1-x)][1 - 4rx(1-x)] \\ &= 16r^2x[-4rx^3 + 8rx^2 - (1+4r)x + 1]. \end{aligned} \quad (6.7)$$

What happens if we increase  $r$  still further? Eventually the magnitude of the slope of the fixed points of  $f^{(2)}(x)$  exceeds unity and the fixed points of  $f^{(2)}(x)$  become unstable. Now the cycle of  $f$  is period 4, and we can study the stability of the fixed points of the fourth iterate  $f^{(4)}(x) = f^{(2)}(f^{(2)}(x)) = f(f(f(f(x))))$ . These fixed points also eventually bifurcate, and we are led to the phenomena of *period-doubling* that we observed in Problem 6.2.

**GraphicalSolutionApp** implements the graphical analysis of  $f(x)$ . The  $n$ th order iterates are defined in **f(x,r,iterate)** a *recursive* method. (The quantity **iterate** is 1, 2, and 4 for the functions  $f(x)$ ,  $f^{(2)}(x)$ , and  $f^{(4)}(x)$  respectively.) Recursion is an idea that is simple once you understand it, but it can be difficult to grasp the idea initially. One way to understand how recursion works is to think of a stack, such as a stack of trays in a cafeteria. The first time a recursive function is called, the function is placed on the top of the stack. Each time the function calls itself, an exact copy of the function, with possibly different values of the input parameters, is placed on top of the stack. When a copy of the function is finished, this copy is removed from (“popped off”) the stack, just as the top tray of a stack of trays would be removed first.

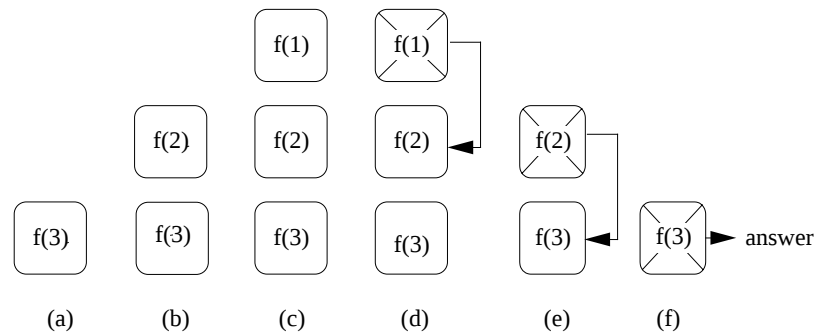


Figure 6.4: Example of the calculation of  $f(0.4, 0.8, 3)$  using the recursive function defined in `GraphicalSolutionApp`. The number in each box is the value of the variable `iterate`. The values of  $x = 0.4$  and  $r = 0.8$  are not shown. The value of  $f(x, r, 3) = 0.7842$ .

To understand the function  $f(x, r, \text{iterate})$ , suppose we want to compute  $f(0.4, 0.8, 3)$ . First we write  $f(0.4, 0.8, 3)$  as in Figure 6.4a. Follow the statements within the function until another call to  $f(0.4, 0.8, \text{iterate})$  occurs. In this case, the call is to  $f(0.4, 0.8, \text{iterate}-1)$  which equals  $f(0.4, 0.8, 2)$ . Write  $f(0.4, 0.8, 2)$  above  $f(0.4, 0.8, 3)$  (see Figure 6.4b). When you come to the end of the definition of the function, write down the value of  $f$  that is actually returned, and remove the function from the stack by crossing it out (see Figure 6.4d). This returned value for  $f$  equals  $y$  if  $\text{iterate} > 1$ , or it is the output of the function for  $\text{iterate} = 1$ . Continue deleting copies of  $f$  as they are finished, until there are no copies left on the paper. The final value of  $f$  is the value returned by the computer. Write a short program that defines  $f(x, r, \text{iterate})$  and prints the value of  $f(0.4, 0.8, 3)$ . Is the answer the same as your hand calculation?

Listing 6.3: `GraphicalSolutionApp` displays the graphical solution of the logistic map trajectory.

```
package org.opensourcephysics.sip.ch06;

import org.opensourcephysics.controls.AbstractSimulation;
import org.opensourcephysics.controls.SimulationControl;
import org.opensourcephysics.frames.PlotFrame;

public class GraphicalSolutionApp extends AbstractSimulation {
    PlotFrame plotFrame = new PlotFrame("iterations", "x", "graphical solution");
    double r; // control parameter
    int iterate; // iterate of f(x)
    double x, y;
    double x0, y0;
    public GraphicalSolutionApp() {
        plotFrame.setPreferredMinMax(0,1,0,1);
        plotFrame.setConnected(true);
        plotFrame.setXPointsLinked(true);
        plotFrame.setMarkerShape(0, 0); // second argument indicates no marker
    }
}
```

```

    plotFrame.setMarkerShape(2, 0); // second argument indicates no marker
}

public void reset() {
    control.setValue("r", 0.89);
    control.setValue("x", 0.2);
    control.setValue("iterate", 1);
}

public void initialize () {
    r = control.getDouble("r");
    x = control.getDouble("x");
    iterate = control.getInt("iterate");
    x0 = x;
    y0 = 0;
    clear ();
}

public void startAnimation() {
    if( iterate!=control.getInt("iterate")) {
        iterate = control.getInt("iterate");
        clear ();
    }
    r = control.getDouble("r");
    super.startAnimation();
}

public void stepAnimation() {
    if( iterate!=control.getInt("iterate")) {
        iterate = control.getInt("iterate");
        clear ();
    }
    r = control.getDouble("r");
    super.stepAnimation();
}

public void doStep() {
    y = f(x, r, iterate );
    plotFrame.append(1, x0, y0);
    plotFrame.append(1, x0, y);
    plotFrame.append(1, y, y);
    x = x0 = y0 = y;
    control.setValue("x", x);
}

void drawFunction() {
    int nplot = 200; // # of points at which function computed
    double delta = 1.0/nplot;
    double x = 0;
    double y = 0;

```

```

    for(int i = 0; i <= nplot; i++) {
        y = f(x, r, iterate);
        plotFrame.append(0, x, y);
        x += delta;
    }
}

void drawLine() { // draws line y = x
    for(double x = 0; x < 1; x += 0.001) {
        plotFrame.append(2, x, x);
    }
}

public double f(double x, double r, int iterate) {
    if( iterate > 1) {
        double y = f(x, r, iterate - 1);
        return 4*r*y*(1-y);
    } else {
        return 4*r*x*(1-x);
    }
}

public void clear() {
    plotFrame.clearData();
    drawFunction();
    drawLine();
    plotFrame.repaint();
}

public static void main(String[] args) {
    SimulationControl control = SimulationControl.createApp(new GraphicalSolutionApp());
    control.addButton("clear", "Clear", "Clears the trajectory.");
}
}

```

**Problem 6.4.** Qualitative properties of the fixed points

- a. Use `GraphicalSolutionApp` to show graphically that there is a single stable fixed point of  $f(x)$  for  $r < 3/4$ . It would be instructive to modify the program so that the value of the slope  $y_n = f(x_n)$  is shown at each iteration as you step each iteration. At what value of  $r$  does the absolute value of this slope exceed unity? Let  $b_1$  denote the value of  $r$  at which the fixed point of  $f(x)$  bifurcates and becomes unstable. Verify that  $b_1 = 0.75$ .
- b. Describe the trajectory of  $f(x)$  for  $r = 0.785$ . What is the nature of the fixed point given by  $x = 1 - 1/4r$ ? What is the nature of the trajectory if  $x_0 = 1 - 1/4r$ ? What is the period of  $f(x)$  for all other choices of  $x_0$ ? What are the values of the two-point attractor?
- c. The function  $f(x)$  is symmetrical about  $x = 1/2$  where  $f(x)$  is a maximum. What are the qualitative features of the second iterate  $f^{(2)}(x)$  for  $r = 0.785$ ? Is  $f^{(2)}(x)$  symmetrical about

- $x = 1/2$ ? For what value of  $x$  does  $f^{(2)}(x)$  have a minimum? Iterate  $x_{n+1} = f^{(2)}(x_n)$  for  $r = 0.785$  and find its two fixed points  $x_1^*$  and  $x_2^*$ . (Try  $x_0 = 0.1$  and  $x_0 = 0.3$ .) Are the fixed points of  $f^{(2)}(x)$  stable or unstable? How do these values of  $x_1^*$  and  $x_2^*$  compare with the values of the two-point attractor of  $f(x)$ ? Verify that the slopes of  $f^{(2)}(x)$  at  $x_1^*$  and  $x_2^*$  are equal.
- Verify the following properties of the fixed points of  $f^{(2)}(x)$ . As  $r$  is increased, the fixed points of  $f^{(2)}(x)$  move apart and the slope of  $f^{(2)}(x)$  at its fixed points decreases. What is the value of  $r = s_2$  at which one of the two fixed points of  $f^{(2)}$  equals  $1/2$ ? What is the value of the other fixed point? What is the slope of  $f^{(2)}(x)$  at  $x = 1/2$ ? What is the slope at the other fixed point? As  $r$  is further increased, the slopes at the fixed points become negative. Finally at  $r = b_2 \approx 0.8623$ , the slopes at the two fixed points of  $f^{(2)}(x)$  equal  $-1$ , and the two fixed points of  $f^{(2)}$  become unstable. (The exact value of  $b_2$  is  $b_2 = (1 + \sqrt{6})/4$ .)
  - Show that for  $r$  slightly greater than  $b_2$ , for example,  $r = 0.87$ , there are four stable fixed points of the map  $f^{(4)}(x)$ . What is the value of  $r = s_3$  when one of the fixed points equals  $1/2$ ? What are the values of the three other fixed points at  $r = s_3$ ?
  - Estimate the value of  $r = b_3$  at which the four fixed points of  $f^{(4)}$  become unstable.
  - Choose  $r = s_3$  and estimate the number of iterations that are necessary for the trajectory to converge to period 4 behavior. How does this number of iterations change when neighboring values of  $r$  are considered? Choose several values of  $x_0$  so that your results do not depend on the initial conditions.

**Problem 6.5.** Periodic windows in the chaotic regime

- If you look closely at the bifurcation diagram in Figure 6.2, you will see that the range of chaotic behavior for  $r > r_\infty$  is interrupted by intervals of periodic behavior. Magnify your bifurcation diagram so that you can look at the interval  $0.957107 \leq r \leq 0.960375$ , where a periodic trajectory of period 3 occurs. (Period 3 behavior starts at  $r = (1 + \sqrt{8})/4$ .) What happens to the trajectory for slightly larger  $r$ , for example, for  $r = 0.9604$ ?
- Plot the map  $f^{(3)}(x)$  versus  $x$  at  $r = 0.96$ , a value of  $r$  in the period 3 window. Draw the line  $y = x$  and determine the intersections with  $f^{(3)}(x)$ . The stable fixed points satisfy the condition  $x^* = f^{(3)}(x^*)$ . Because  $f^{(3)}(x)$  is an eighth-order polynomial, there are eight solutions (including  $x = 0$ ). Find the intersections of  $f^{(3)}(x)$  with  $y = x$  and identify the three stable fixed points. What are the slopes of  $f^{(3)}(x)$  at these points? Then decrease  $r$  to  $r = 0.957107$ , the (approximate) value of  $r$  below which the system is chaotic. Draw the line  $y = x$  and determine the number of intersections with  $f^{(3)}(x)$ . Note that at this value of  $r$ , the curve  $y = f^{(3)}(x)$  is tangent to the diagonal line at the three stable fixed points. For this reason, this type of transition is called a *tangent bifurcation*. Note that there also is an unstable point at  $x \approx 0.76$ .
- Plot  $x_{n+1} = f^{(3)}(x_n)$  versus  $n$  for  $r = 0.9571$ , a value of  $r$  just below the onset of period 3 behavior. How would you describe the behavior of the trajectory? This type of chaotic motion is an example of *intermittency*, that is, nearly periodic behavior interrupted by occasional irregular bursts.

- d. To understand the mechanism behind the intermittent behavior, we need to “zoom in” on the values of  $x$  near the stable fixed points that you found in part (c). To do so you can change the arguments of the `setPreferredMinMax` method. If you zoom in enough, you will see a narrow channel between the diagonal line  $y = x$  and the plot of  $f^{(3)}(x)$  near each fixed point. The trajectory can require many iterations to squeeze through the channel, and we see period 3 behavior during this time. Eventually, the trajectory escapes from the channel and bounces around until it is again sent into a channel at some unpredictable later time.

## 6.4 Universal Properties and Self-Similarity

In Sections 6.2 and 6.3 we found that the trajectory of the logistic map has remarkable properties as a function of the control parameter  $r$ . In particular, we found a sequence of period-doublings accumulating in a chaotic trajectory of infinite period at  $r = r_\infty$ . For most values of  $r > r_\infty$ , the trajectory is very sensitive to the initial conditions. We also found “windows” of period 3, 6, 12, ... embedded in the range of chaotic behavior. How typical is this type of behavior? In the following, we will find further numerical evidence that the general behavior of the logistic map is independent of the details of the form (6.5) of  $f(x)$ .

You might have noticed that the range of  $r$  between successive bifurcations becomes smaller as the period increases (see Table 6.1). For example,  $b_2 - b_1 = 0.112398$ ,  $b_3 - b_2 = 0.023624$ , and  $b_4 - b_3 = 0.00508$ . A good guess is that the decrease in  $b_k - b_{k-1}$  is geometric, that is, the ratio  $(b_k - b_{k-1})/(b_{k+1} - b_k)$  is a constant. You can check that this ratio is not exactly constant, but converges to a constant with increasing  $k$ . This behavior suggests that the sequence of values of  $b_k$  has a limit and follows a geometrical progression:

$$b_k \approx r_\infty - C \delta^{-k}, \quad (6.8)$$

where  $\delta$  is known as the *Feigenbaum number* and  $C$  as a constant. From (6.8) it is easy to show that  $\delta$  is given by the ratio

$$\delta = \lim_{k \rightarrow \infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k}. \quad (6.9)$$

**Problem 6.6.** Estimation of the Feigenbaum constant

- a. Plot  $\delta_k = (b_k - b_{k-1})/(b_{k+1} - b_k)$  versus  $k$  using the values of  $b_k$  in Table 6.1 and estimate the value of  $\delta$ . Are the number of decimal places given in Table 6.1 for  $b_k$  sufficient for all the values of  $k$  shown? The best estimate of  $\delta$  is

$$\delta = 4.669\,201\,609\,102\,991\dots \quad (6.10)$$

The number of decimal places in (6.10) is shown to indicate that  $\delta$  is known precisely. Use (6.8) and (6.10) and the values of  $b_k$  to estimate the value of  $r_\infty$ .

- b. In Problem 6.4 we found that one of the four fixed points of  $f^{(4)}(x)$  is at  $x^* = 1/2$  for  $r = s_3 \approx 0.87464$ . We also found that the convergence to the fixed points of  $f^{(4)}(x)$  for this value of  $r$  is more rapid than at nearby values of  $r$ . In Appendix 6A we show that these *superstable*

| $k$ | $b_k$     |
|-----|-----------|
| 1   | 0.750 000 |
| 2   | 0.862 372 |
| 3   | 0.886 023 |
| 4   | 0.891 102 |
| 5   | 0.892 190 |
| 6   | 0.892 423 |
| 7   | 0.892 473 |
| 8   | 0.892 484 |

Table 6.1: Values of the control parameter  $r = b_k$  for the onset of the  $k$ th bifurcation. Six decimal places are shown.

trajectories occur whenever one of the fixed points is at  $x = 1/2$ . The values of  $r = s_m$  that give superstable trajectories of period  $2^{m-1}$  are much better defined than the points of bifurcation,  $r = b_k$ . The rapid convergence to the final trajectories also gives better numerical estimates, and we always know one member of the trajectory, namely  $x = 1/2$ . It is reasonable that  $\delta$  can be defined as in (6.9) with  $b_k$  replaced by  $s_m$ . Use  $s_1 = 0.5$ ,  $s_2 \approx 0.809017$ , and  $s_3 = 0.874640$  to estimate  $\delta$ . The numerical values of  $s_m$  are found in Project 6.22 by solving the equation  $f^{(m)}(x = 1/2) = 1/2$  numerically; the first eight values of  $s_m$  are listed in Table 6.2 in Section 6.11.

We can associate another number with the series of pitchfork bifurcations. From Figure 6.3 and Figure 6.5 we see that each pitchfork bifurcation gives birth to “twins” with the new generation more densely packed than the previous generation. One measure of this density is the maximum distance  $M_k$  between the values of  $x$  describing the bifurcation (see Figure 6.5). The disadvantage of using  $M_k$  is that the transient behavior of the trajectory is very long at the boundary between two different periodic behaviors. A more convenient measure of the density is the quantity  $d_k = x_k^* - 1/2$ , where  $x_k^*$  is the value of the fixed point nearest to the fixed point  $x^* = 1/2$ . The first two values of  $d_k$  are shown in Figure 6.6 with  $d_1 \approx 0.3090$  and  $d_2 \approx -0.1164$ . The next value is  $d_3 \approx 0.0460$ . Note that the fixed point nearest to  $x = 1/2$  alternates from one side of  $x = 1/2$  to the other. We define the quantity  $\alpha$  by the ratio

$$\alpha = \lim_{k \rightarrow \infty} -\left(\frac{d_k}{d_{k+1}}\right). \quad (6.11)$$

The estimates  $\alpha = 0.3090/0.1164 = 2.65$  for  $k = 1$  and  $\alpha = 0.1164/0.0460 = 2.53$  for  $k = 2$  are consistent with the asymptotic value  $\alpha = 2.5029078750958928485 \dots$

We now give qualitative arguments that suggest that the general behavior of the logistic map in the period-doubling regime is independent of the detailed form of  $f(x)$ . As we have seen, period-doubling is characterized by self-similarities, for example, the period-doublings look similar except for a change of scale. We can demonstrate these similarities by comparing  $f(x)$  for  $r = s_1 = 0.5$  for the superstable trajectory with period 1 to the function  $f^{(2)}(x)$  for  $r = s_2 \approx 0.809017$  for the superstable trajectory of period 2 (see Figure 6.7). The function  $f(x, r = s_1)$  has unstable fixed points at  $x = 0$  and  $x = 1$  and a stable fixed point at  $x = 1/2$ . Similarly the function  $f^{(2)}(x, r = s_2)$

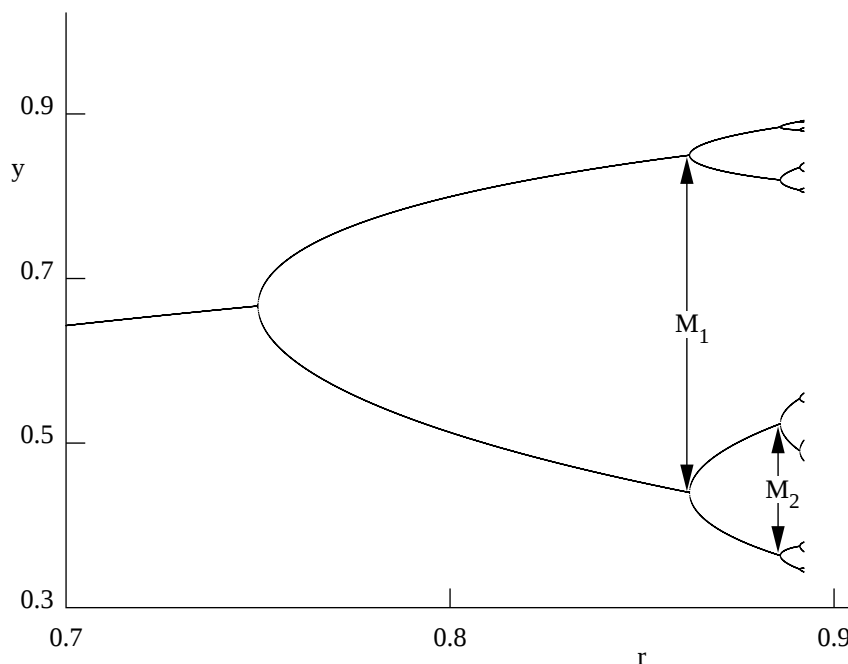


Figure 6.5: The first few bifurcations of the logistic equation showing the scaling of the maximum distance  $M_k$  between the asymptotic values of  $x$  describing the bifurcation.

has a stable fixed point at  $x = 1/2$  and an unstable fixed point at  $x \approx 0.69098$ . Note the similar shape, but different scale of the curves in the square box in part (a) and part (b) of Figure 6.7. This similarity is an example of scaling. That is, if we scale  $f^{(2)}$  and change (*renormalize*) the value of  $r$ , we can compare  $f^{(2)}$  to  $f$ . (See Chapter 12 for a discussion of scaling and renormalization in another context.)

This graphical comparison is meant only to be suggestive. A precise approach shows that if we continue the comparison of the higher-order iterates, for example,  $f^{(4)}(x)$  to  $f^{(2)}(x)$ , etc., the superposition of functions converges to a universal function that is independent of the form of the original function  $f(x)$ .

**Problem 6.7.** Further estimates of the exponents  $\alpha$  and  $\delta$

1. Write a method to find the appropriate scaling factor and superimpose  $f$  and the rescaled form of  $f^{(2)}$  found in Figure 6.7.
2. Use arguments similar to those discussed in the text in Figure 6.7 and compare the behavior of  $f^{(4)}(x, r = s_3)$  in the square about  $x = 1/2$  with  $f^{(2)}(x, r = s_2)$  in its square about  $x = 1/2$ . The size of the squares are determined by the unstable fixed point nearest to  $x = 1/2$ . Find the appropriate scaling factor and superimpose  $f^{(2)}$  and the rescaled form of  $f^{(4)}$ .

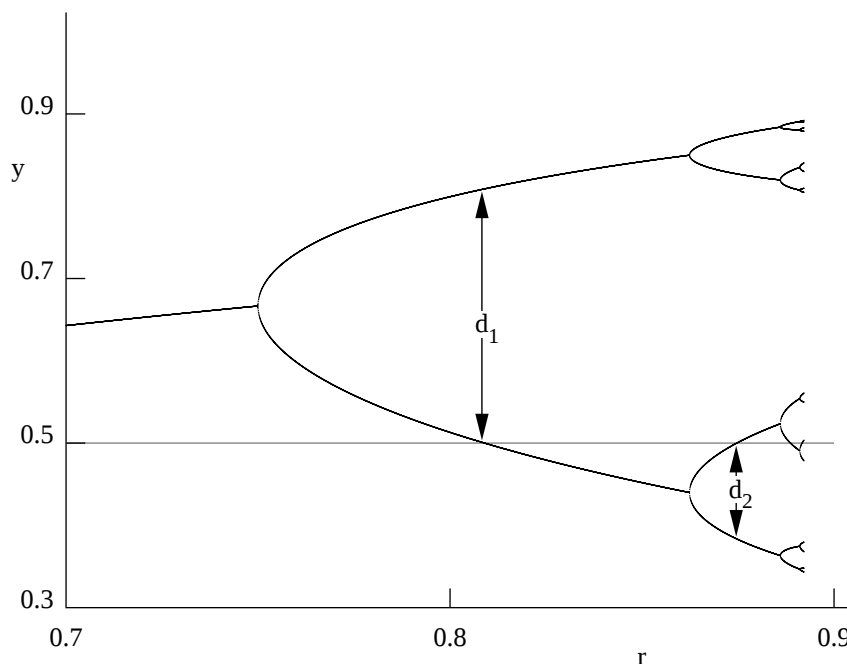


Figure 6.6: The quantity  $d_k$  is the distance from  $x^* = 1/2$  to the nearest element of the attractor of period  $2^k$ . It is convenient to use this quantity to determine the exponent  $\alpha$ .

It is easy to modify your programs to consider other one-dimensional maps. In Problem 6.8 we consider several one-dimensional maps and determine if they also exhibit the period-doubling route to chaos.

**\*Problem 6.8.** Other one-dimensional maps

Determine the qualitative properties of the one-dimensional maps:

$$f(x) = xe^{r(1-x)} \quad (6.12)$$

$$f(x) = r \sin \pi x. \quad (6.13)$$

The map in (6.12) has been used by ecologists (cf. May) to study a population that is limited at high densities by the effect of epidemic disease. Although it is more complicated than (6.5), its advantage is that the population remains positive no matter what (positive) value is taken for the initial population. There are no restrictions on the maximum value of  $r$ , but if  $r$  becomes sufficiently large,  $x$  eventually becomes effectively zero, rendering the population extinct. What is the behavior of the time series of (6.12) for  $r = 1.5, 2$ , and  $2.7$ ? Describe the qualitative behavior of  $f(x)$ . Does it have a maximum?

The sine map (6.13) with  $0 < r \leq 1$  and  $0 \leq x \leq 1$  has no special significance, except that it is nonlinear. If time permits, estimate the value of  $\delta$  for both maps. What limits the accuracy of your determination of  $\delta$ ?

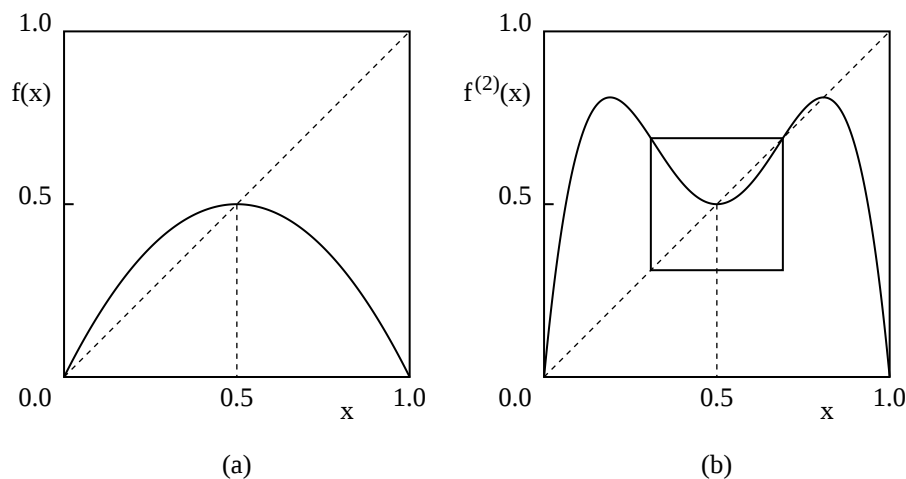


Figure 6.7: Comparison of  $f(x, r)$  for  $r = s_1$  with the second iterate  $f^{(2)}(x)$  for  $r = s_2$ . (a) The function  $f(x, r = s_1)$  has unstable fixed points at  $x = 0$  and  $x = 1$  and a stable fixed point at  $x = 1/2$ . (b) The function  $f^{(2)}(x, r = s_1)$  has a stable fixed point at  $x = 1/2$ . The unstable fixed point of  $f^{(2)}(x)$  nearest to  $x = 1/2$  occurs at  $x \approx 0.69098$ , where the curve  $f^{(2)}(x)$  intersects the line  $y = x$ . The upper right-hand corner of the square box in (b) is located at this point, and the center of the box is at  $(1/2, 1/2)$ . Note that if we reflect this square about the point  $(1/2, 1/2)$ , the shape of the reflected graph in the square box is nearly the same as it is in part (a), but on a smaller scale.

The above qualitative arguments and numerical results suggest that the quantities  $\alpha$  and  $\delta$  are *universal*, that is, independent of the detailed form of  $f(x)$ . In contrast, the values of the accumulation point  $r_\infty$  and the constant  $C$  in (6.8) depend on the detailed form of  $f(x)$ . Feigenbaum has shown that the period-doubling route to chaos and the values of  $\delta$  and  $\alpha$  are universal property of maps that have a quadratic maximum, that is,  $f'(x)|_{x=x_m} = 0$  and  $f''(x)|_{x=x_m} < 0$ .

Why is the universality of period-doubling and the numbers  $\delta$  and  $\alpha$  more than a curiosity? The reason is that because this behavior is independent of the details, there might exist realistic systems whose underlying dynamics yield the same behavior as the logistic map. Of course, most physical systems are described by differential rather than difference equations. Can these systems exhibit period-doubling behavior? Several workers (cf. Testa et al.) have constructed nonlinear RLC circuits driven by an oscillatory source voltage. The output voltage shows bifurcations, and the measured values of the exponents  $\delta$  and  $\alpha$  are consistent with the predictions of the logistic map.

Of more general interest is the nature of turbulence in fluid systems. Consider a stream of water flowing past several obstacles. We know that at low flow speeds, the water flows past obstacles in a regular and time-independent fashion, called *laminar* flow. As the flow speed is increased (as measured by a dimensionless parameter called the Reynolds number), some swirls develop, but the motion is still time-independent. As the flow speed is increased still further, the swirls break away and start moving downstream. The flow pattern as viewed from the bank

becomes time-dependent. For still larger flow speeds, the flow pattern becomes very complex and looks random. We say that the flow pattern has made a transition from laminar flow to *turbulent* flow.

This qualitative description of the transition to chaos in fluid systems is superficially similar to the description of the logistic map. Can fluid systems be analyzed in terms of the simple models of the type we have discussed here? In a few instances such as turbulent convection in a heated saucepan, period doubling and other types of transitions to turbulence have been observed. The type of theory and analysis we have discussed has suggested new concepts and approaches, and the study of turbulent flows is a subject of much current interest.

## 6.5 Measuring Chaos

How do we know if a system is chaotic? The most important characteristic of chaos is *sensitivity to initial conditions*. In Problem 6.3 for example, we found that the trajectories starting from  $x_0 = 0.5$  and  $x_0 = 0.5001$  for  $r = 0.91$  become very different after a small number of iterations. Because computers only store floating numbers to a certain number of digits, the implication of this result is that our numerical predictions of the trajectories of chaotic systems are restricted to small time intervals. That is, sensitivity to initial conditions implies that even though the logistic map is deterministic, our ability to make numerical predictions of its trajectory is limited.

How can we quantify this lack of predictability? In general, if we start two identical dynamical systems from slightly different initial conditions, we expect that the difference between the trajectories will change as a function of  $n$ . In Figure 6.8 we show a plot of the difference  $|\Delta x_n|$  versus  $n$  for the same conditions as in Problem 6.3a. We see that roughly speaking,  $\ln |\Delta x_n|$  is a linearly increasing function of  $n$ . This result indicates that the separation between the trajectories grows exponentially if the system is chaotic. This divergence of the trajectories can be described by the *Lyapunov* exponent  $\lambda$ , which is defined by the relation:

$$|\Delta x_n| = |\Delta x_0| e^{\lambda n}, \quad (6.14)$$

where  $\Delta x_n$  is the difference between the trajectories at time  $n$ . If the Lyapunov exponent  $\lambda$  is positive, then nearby trajectories diverge exponentially. Chaotic behavior is characterized by the *exponential divergence of nearby trajectories*.

A naive way of measuring the Lyapunov exponent  $\lambda$  is to run the same dynamical system twice with slightly different initial conditions and measure the difference of the trajectories as a function of  $n$ . We used this method to generate Figure 6.8. Because the rate of separation of the trajectories might depend on the choice of  $x_0$ , a better method would be to compute the rate of separation for many values of  $x_0$ . This method would be tedious, because we would have to fit the separation to (6.14) for each value of  $x_0$  and then determine an average value of  $\lambda$ .

A more important limitation of the naive method is that because the trajectory is restricted to the unit interval, the separation  $|\Delta x_n|$  ceases to increase when  $n$  becomes sufficiently large. Fortunately, there is a better way of determining  $\lambda$ . To understand it, we take the natural logarithm of both sides of (6.14), and write  $\lambda$  as

$$\lambda = \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|. \quad (6.15)$$

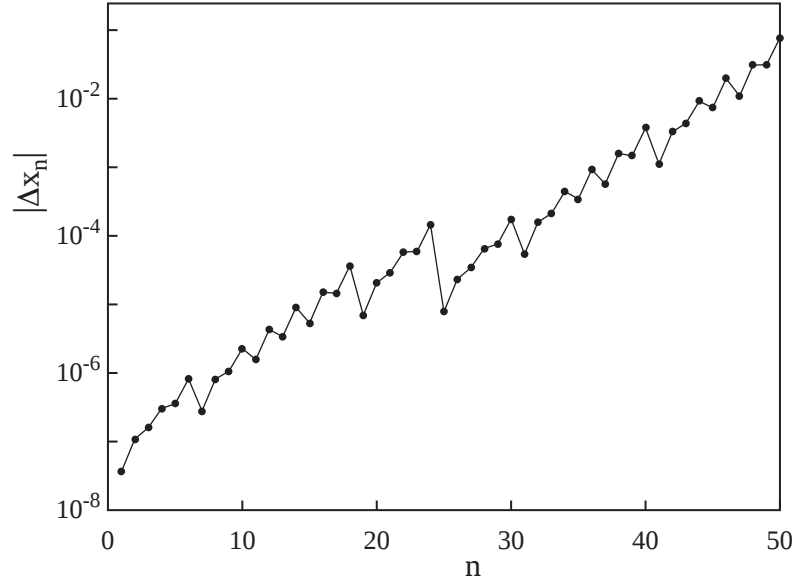


Figure 6.8: The evolution of the difference  $\Delta x_n$  between the trajectories of the logistic map at  $r = 0.91$  for  $x_0 = 0.5$  and  $x_0 = 0.5001$ . The separation between the two trajectories increases with  $n$ , the number of iterations, if  $n$  is not too large. (Note that  $|\Delta x_1| \sim 10^{-8}$  and that the trend is not monotonic.)

Because we want to use the data from the entire trajectory after the transient behavior has ended, we use the fact that

$$\frac{\Delta x_n}{\Delta x_0} = \frac{\Delta x_1}{\Delta x_0} \frac{\Delta x_2}{\Delta x_1} \dots \frac{\Delta x_n}{\Delta x_{n-1}}. \quad (6.16)$$

Hence, we can express  $\lambda$  as

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right|. \quad (6.17)$$

The form (6.17) implies that we can interpret  $x_i$  for any  $i$  as the initial condition.

We see from (6.17) that the problem of computing  $\lambda$  has been reduced to finding the ratio  $\Delta x_{i+1}/\Delta x_i$ . Because we want to make the initial difference between the two trajectories as small as possible, we are interested in the limit  $\Delta x_i \rightarrow 0$ .

The idea of the more sophisticated procedure is to compute the differential  $dx_i$  from the equation of motion at the same time that the equation of motion is being iterated. We use the logistic map as an example. The differential of (6.5) can be written as

$$\frac{dx_{i+1}}{dx_i} = f'(x_i) = 4r(1 - 2x_i). \quad (6.18)$$

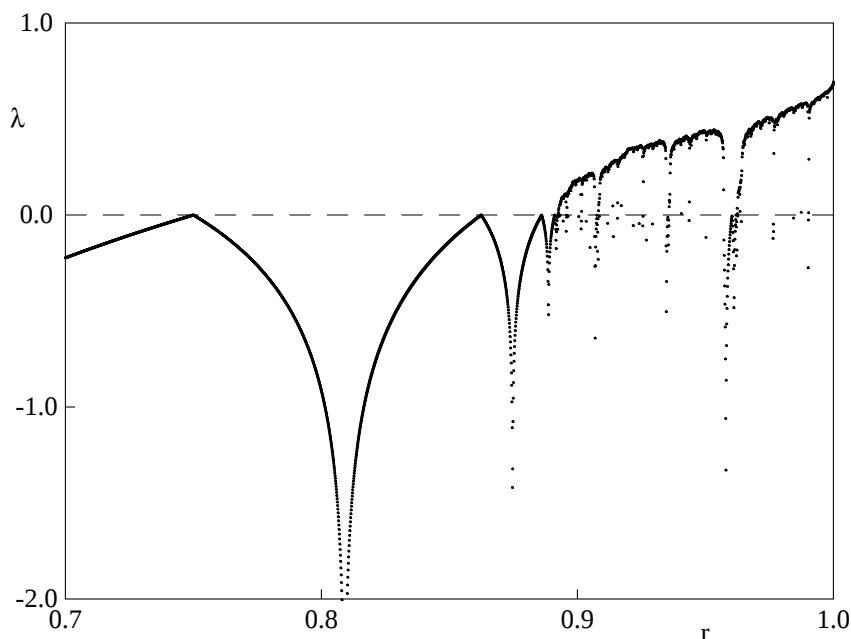


Figure 6.9: The Lyapunov exponent calculated using the method in (6.19) as a function of the control parameter  $r$ . Compare the behavior of  $\lambda$  to the bifurcation diagram in Figure 6.2. Note that  $\lambda < 0$  for  $r < 3/4$  and approaches zero at a period doubling bifurcation. A negative spike corresponds to a superstable trajectory. The onset of chaos is visible near  $r = 0.892$ , where  $\lambda$  first becomes positive. For  $r > 0.892$ ,  $\lambda$  generally increases except for dips below zero whenever a periodic window occurs. Note the dip due to the period 3 window near  $r = 0.96$ . For each value of  $r$ , the first 1000 iterations were discarded, and  $10^5$  values of  $\ln |f'(x_n)|$  were used to determine  $\lambda$ .

We can consider  $x_i$  for any  $i$  as the initial condition and the ratio  $dx_{i+1}/dx_i$  as a measure of the rate of change of  $x_i$ . Hence, we can iterate the logistic map as before and use the values of  $x_i$  and the relation (6.18) to compute  $dx_{i+1}/dx_i$  at each iteration. The Lyapunov exponent is given by

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|, \quad (6.19)$$

where we begin the sum in (6.19) after the transient behavior is finished. We have included explicitly the limit  $n \rightarrow \infty$  in (6.19) to remind ourselves to choose  $n$  sufficiently large. Note that this procedure weights the points on the attractor correctly, that is, if a particular region of the attractor is not visited often by the trajectory, it does not contribute much to the sum in (6.19).

**Problem 6.9.** Lyapunov exponent for the logistic map

- Modify `BifurcateApp` and first compute the Lyapunov exponent  $\lambda$  for the logistic map using the naive approach. Choose  $r = 0.91$ ,  $x_0 = 0.5$ , and  $\Delta x_0 = 10^{-6}$ , and plot  $\ln |\Delta x_n / \Delta x_0|$  versus

- n.* What happens to  $\ln|\Delta x_n/\Delta x_0|$  for large  $n$ ? Estimate  $\lambda$  for  $r = 0.91$ ,  $r = 0.97$ , and  $r = 1.0$ . Does your estimate of  $\lambda$  for each value of  $r$  depend significantly on your choice of  $x_0$  or  $\Delta x_0$ ?
- b.* Compute  $\lambda$  using the algorithm discussed in the text for  $r = 0.76$  to  $r = 1.0$  in steps of  $\Delta r = 0.01$ . What is the sign of  $\lambda$  if the system is not chaotic? Plot  $\lambda$  versus  $r$ , and explain your results in terms of behavior of the bifurcation diagram shown in Figure 6.2. Compare your results for  $\lambda$  with those shown in Figure 6.9. How does the sign of  $\lambda$  correlate with the behavior of the system as seen in the bifurcation diagram? If  $\lambda < 0$ , the two trajectories converge and the system is not chaotic. If  $\lambda = 0$ , then the trajectories diverge algebraically, that is, as a power of  $n$ , as will show in the following. For what value of  $r$  is  $\lambda$  a maximum?
- c.* In Problem 6.3b we saw that roundoff errors in the chaotic regime make the computation of individual trajectories meaningless. That is, if the system's behavior is chaotic, then small roundoff errors are amplified exponentially in time, and the actual numbers we compute for the trajectory starting from a given initial value are not "real." Given this limitation, how meaningful is our computation of the Lyapunov exponent? Repeat your calculation of  $\lambda$  for  $r = 1$  by changing the roundoff error as you did in Problem 6.3b. Does your computed value of  $\lambda$  change? We will encounter a similar question in Chapter 8 where we compute the trajectories of a system of many particles. The answer appears to be that although the trajectory we compute is not the one we thought we were trying to compute, the computed trajectory is close to a possible trajectory of the system. Quantities such as  $\lambda$  that are averaged over many possible trajectories are independent of the detailed behavior of an individual trajectory.

In some cases a dynamical system is at the "edge of chaos" where the Lyapunov exponent vanishes. Such systems are said to exhibit weak chaos to distinguish their behavior from the strongly chaotic behavior that we have been discussing. In strong chaos two nearby orbits will diverge exponentially as shown in (6.14). If we define  $z \equiv |\Delta x_n|/|\Delta x_0|$ , then  $z$  will satisfy the differential equation

$$\frac{dz}{dn} = \lambda z. \quad (6.20)$$

For weak chaos we do not find an exponential divergence, but instead a divergence that is algebraic and is given by

$$\frac{dz}{dn} = \lambda_q z^q, \quad (6.21)$$

where  $q$  is a parameter that needs to be determined. The solution to (6.21) is

$$z = [1 + (1 - q)\lambda_q n]^{1/(1-q)}, \quad (6.22)$$

which can be checked by substituting (6.22) into (6.21). In the limit  $q \rightarrow 1$ , we recover the usual exponential dependence.

We can determine the type of chaos using the crude approach of choosing a large number of initial values of  $x_0$  and  $x_0 + \Delta x_0$  and plotting the average of  $\ln z$  versus  $n$ . If we do not obtain a straight line, then the system does not exhibit strong chaos. How can we check for the behavior shown in (6.22)? The easiest way is to plot the function

$$\frac{z^{1-q} - 1}{1 - q}, \quad (6.23)$$

which will equal  $n\lambda_q$  if (6.22) is applicable. We explore these ideas in the following problem.

**\*Problem 6.10.** Measuring weak chaos

- a. Write a program that plots  $\ln z$  if  $q = 1$  or  $z_q$  if  $q \neq 1$  as a function of  $n$ . Your program should have  $q$ ,  $|\Delta x_0|$ , the number of seeds, and the number of iterations as input parameters. To compare with published work by Añãños and Tsallis, use a variation of the logistic map given by

$$x_{n+1} = 1 - ax_n^2, \quad (6.24)$$

where  $-1 \leq x_n \leq 1$  and  $0 \leq a \leq 2$ . The edge of chaos is at  $a = 1.401155189$ . The seeds  $x_0$  should be equally spaced in the interval  $-1 < x_0 < 1$ .

- b. Consider strong chaos at  $a = 2$ . Choose  $q = 1$ , 50 iterations, and at least 1000 values of  $x_0$ . Try  $|\Delta x_0| = 10^{-6}$ . Do you obtain a straight line for  $\ln z$  versus  $n$ ? Does the curve eventually stop growing? If so why? Try  $|\Delta x_0| = 10^{-12}$ . How do your results differ and how are they the same? If you run into problems, you may have lost too much precision. To overcome this problem, also iterate  $\Delta x$  directly:

$$\Delta x_{n+1} = x_{n+1} - \tilde{x}_{n+1} = -a(x_n^2 - \tilde{x}_n^2) = -a(x_n - \tilde{x}_n)(x_n + \tilde{x}_n) = -a\Delta x_n(x_n + \tilde{x}_n), \quad (6.25)$$

where  $x_n$  is the iterate starting at  $x_0$  and  $\tilde{x}_n$  is the iterate starting at  $x_0 + \Delta x_0$ . Show that straight lines are not obtained for your plot if  $q \neq 1$ .

- c. Repeat part (a) with  $a = 1.401155189$  and various values of  $q$ . Simulations with  $10^5$  starting points show that linear behavior is obtained for  $q \approx 0.36$ .

In the study of many particle systems it is assumed that a system of fixed energy (and number of particles and volume) has an equal probability of being in any microstate specified by the positions and velocities of the particles (see Sec 15.2). This idea is closely related to chaos because the particles evolve deterministically, yet they are expected to behave as if they are random. One way of measuring the ability of a system to be in any state is to measure its entropy defined by

$$S = - \sum_i p_i \ln p_i, \quad (6.26)$$

where the sum is over all states and  $p_i$  is the probability or relative frequency of being in the  $i$ th state. For example, if the system is always in only one state, then  $S = 0$ , the smallest possible entropy. If the system explores all states equally, then  $S = \ln \Omega$ , where  $\Omega$  is the number of possible states. (You can show this result by letting  $p_i = 1/\Omega$ .)

**\*Problem 6.11.** Entropy of the logistic map

- a. We can determine  $S$  for the logistic map by dividing the interval  $[0, 1]$  into  $\Omega$  bins or subintervals and determining the relative number of times the trajectory falls into each bin. Write a program to compute  $S$  and determine its dependence on  $r$  in the range  $0.7 \leq r \leq 1$ . At each value of  $r$  the map should be iterated for a fixed number of steps, for example,  $n = 1000$ . Choose a bin width of  $\Delta x = 0.01$ . What happens to the entropy when the trajectory is chaotic?

- b. Repeat part (a) with  $n = 10000$ . For what values of  $r$  does the entropy change significantly? Decrease  $\Delta x$  to 0.001 and repeat. Does this decrease make a difference?
- c. Plot  $p_i$  as a function of  $x$  for  $r = 1$ . For what value(s) of  $x$  is the plot a maximum?

We also can measure the (generalized) entropy as a function of time for short times. As will be seen in Problem 6.12, for strong chaos  $S(n)$  increases linearly with  $n$  until all the possible states are visited. However, for weak chaos this behavior is not found. We can generalize the entropy to a  $q$ -dependent function defined by

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1}, \quad (6.27)$$

where in the limit  $q \rightarrow 1$  we find  $S_q \rightarrow S$ . The following problem discusses measuring the entropy for the same system as in Problem 6.10.

**\*Problem 6.12.** Entropy of weak and strong chaotic systems

- a. Write a program that iterates the map (6.24) and plots  $S$  if  $q = 1$  or  $S_q$  if  $q \neq 1$  as a function of  $n$ . The input parameters should be  $q$ , the number of random seeds in a single bin, the number of bins, and the number of iterations. At each iteration compute the entropy as a function of  $n$ . Then average  $S$  over many trials corresponding to a randomly chosen initial interval for the seeds.
- b. Consider strong chaos at  $a = 2$ . Choose  $q = 1$ , 20 iterations, at least 1000 intervals, and ten times as many starting points. Do you obtain a straight line for  $S$  versus  $n$ ? Does the curve eventually stop growing? If so why? Vary the number of intervals. How do your results differ and how are they the same? Show that straight lines are not obtained for your plot if  $q \neq 1$ .
- c. Repeat part (a) with  $a = 1.401155189$  and various values of  $q$ . Simulations with  $10^5$  intervals show that linear behavior is obtained for  $q \approx 0.36$ , the same value as for the measurements in Problem 6.10.

## 6.6 \*Controlling Chaos

The dream of classical physics was that if the initial conditions and all the forces acting on a system were known, then we could predict the future with as much precision as we desire.

The existence of chaos has shattered that dream. However, if a system is chaotic, we still might be able to control its behavior with small, but carefully chosen perturbations of the system. We will illustrate the method for the logistic map. The application of the method to other one-dimensional systems is straightforward, but the extension to higher dimensional systems is more complicated (cf. Ott, Lai).

Suppose that we want the trajectory to be periodic even though the parameter  $r$  is in the chaotic regime. How can we make the trajectory have periodic behavior without drastically changing  $r$  or imposing an external perturbation that is so large that the internal dynamics of the map

become irrelevant? The key to the solution is that for any value of  $r$  in the chaotic regime, there is an infinite number of trajectories that have unstable periods. This property of the chaotic regime means that if we choose the value of the seed  $x_0$  to be precisely equal to a point on an unstable trajectory with period  $p$ , the subsequent trajectory will have this period. However, if we choose a value of  $x_0$  that differs ever so slightly from this special value, the trajectory will not be periodic. Our goal is to make slight perturbations to the system to keep it on the desired unstable periodic trajectory.

The first step is to find the values of  $x(i)$ ,  $i = 1$  to  $p$ , that constitute the unstable periodic trajectory. It is an interesting numerical problem to find the values of  $x(i)$ , and we consider this problem first. The trick is to find a fixed point of the map  $f^{(p)}$ . That is, we need to find the value of  $x^*$  such that

$$g^{(p)}(x^*) \equiv f^{(p)}(x^*) - x^* = 0. \quad (6.28)$$

The algorithms for finding the solution to (6.28) are called root finding algorithms. You might have heard of Newton's method, which we describe in Appendix 6B. Here we use the simplest root-finding algorithm, the *bisection* method. The algorithm works as follows:

- a. Choose two values,  $x_{\text{left}}$  and  $x_{\text{right}}$ , with  $x_{\text{left}} < x_{\text{right}}$ , such that the product  $g^{(p)}(x_{\text{left}})g^{(p)}(x_{\text{right}}) < 0$ . There must be a value of  $x$  such that  $g^{(p)}(x) = 0$  in the interval  $[x_{\text{left}}, x_{\text{right}}]$ .
- b. Choose the midpoint,  $x_{\text{mid}} = x_{\text{left}} + \frac{1}{2}(x_{\text{right}} - x_{\text{left}}) = \frac{1}{2}(x_{\text{left}} + x_{\text{right}})$ , as the guess for  $x^*$ .
- c. If  $g^{(p)}(x_{\text{mid}})$  has the same sign as  $g^{(p)}(x_{\text{left}})$ , then replace  $x_{\text{left}}$  by  $x_{\text{mid}}$ ; otherwise, replace  $x_{\text{right}}$  by  $x_{\text{mid}}$ . The interval for the location of the root is now reduced.
- d. Repeat steps 2 and 3 until the desired level of precision is achieved.

The following program implements this algorithm for the logistic map. An alternative implementation named `FixedPointApp` that does not use recursion is not shown, but is available in the Chapter 6 package. One possible problem is that some of the roots of  $g^{(p)}(x) = 0$  also are roots of  $g^{(p')}(x) = 0$  for  $p'$  equal to a factor of  $p$ . (For example, if  $p = 6$ , 2 and 3 are factors.) As  $p$  increases, it might become more difficult to find a root that is part of a period  $p$  trajectory and not part of a period  $p'$  trajectory.

Listing 6.4: The `RecursiveFixedPointApp` program finds stable and unstable periodic trajectories with the given period using the bisection root finding algorithm.

```
package org.opensourcephysics.sip.ch06;
import org.opensourcephysics.controls.AbstractCalculation;
import org.opensourcephysics.controls.CalculationControl;

public class RecursiveFixedPointApp extends AbstractCalculation {
    double r; // control parameter
    int period;
    double xleft, xright;
    double gleft, gright;
    public void resetCalculation() {
```

```

    control.setValue("r" , 0.8);           // control parameter r
    control.setValue("period" , 2);        // period
    control.setValue("epsilon" , 0.0000001); // desired precision
    control.setValue("xleft" , 0.01);      // guess for xleft
    control.setValue("xright" , 0.99);     // guess for xright
}

public void calculate() {
    double epsilon = control.getDouble("epsilon"); // desired precision
    r = control.getDouble("r");
    period = control.getInt("period");
    xleft = control.getDouble("xleft");
    xright = control.getDouble("xright");
    gleft = map(xleft, r, period) - xleft;
    gright = map(xright, r, period) - xright;
    if (gleft * gright < 0) {
        while(Math.abs(xleft - xright) > epsilon) {
            bisection();
        }
        double x = 0.5 * (xleft + xright);
        control.println("explicit search for period " + period + " behavior");
        control.println(0 + "\t" + x); // result
        for(int i = 1; i <= 2 * period + 1; i++) {
            x = map(x, r, 1);
            control.println(i + "\t" + x);
        }
    } else {
        control.println("range does not enclose a root");
    }
}

public void bisection() {
    // midpoint between xleft and xright
    double xmid = 0.5 * (xleft + xright);
    double gmid = map(xmid, r, period) - xmid;
    if (gmid * gleft > 0) {
        xleft = xmid; // change xleft
        gleft = gmid;
    } else {
        xright = xmid; // change xright
        gright = gmid;
    }
}

double map(double x, double r, double period) {
    if (period > 1) {
        double y = map(x, r, period - 1);
        return 4 * r * y * (1 - y);
    } else {
        return 4 * r * x * (1 - x);
    }
}

```

```

    }
}

public static void main(String[] args) {
    CalculationControl.createApp(new RecursiveFixedPointApp());
}
}

```

**Problem 6.13.** Unstable periodic trajectories for the logistic map

- Test `RecursiveFixedPointApp` for values of  $r$  for which the logistic map has a stable period with  $p = 1$  and  $p = 2$ . Set the desired precision  $\epsilon$  equal to  $10^{-7}$ . Initially use  $x_{\text{left}} = 0.01$  and  $x_{\text{right}} = 0.99$ . Calculate the stable attractor analytically and compare the results of your program with the analytical results.
- Set  $r = 0.95$  and find the periodic trajectories for  $p = 1, 2, 5, 6, 7, 12, 13$ , and  $19$ .
- Modify `RecursiveFixedPointApp` so that  $n_b$ , the number of bisections needed to obtain the unstable trajectory, is listed. Choose three of the cases considered in part (b), and compute  $n_b$  for the precision  $\epsilon$  equal to  $0.01, 0.001, 0.0001$ , and  $0.00001$ . Determine the functional dependence of  $n_b$  on  $\epsilon$ .

Now that we know how to find the values of the unstable periodic trajectories, we discuss an algorithm for stabilizing this period. Suppose that we wish to stabilize the unstable trajectory of period  $p$  for a choice of  $r = r_0$ . The idea is to make small adjustments of  $r = r_0 + \Delta r$  at each iteration so that the difference between the actual trajectory and the target periodic trajectory is small. If the actual trajectory is  $x_n$  and we wish the trajectory to be at  $x(i)$ , we make the next iterate  $x_{n+1}$  equal to  $x(i+1)$  by expanding the difference  $x_{n+1} - x(i+1)$  in a Taylor series and setting the difference to zero to first-order. We have  $x_{n+1} - x(i+1) = f(x_n, r) - f(x(i), r_0)$ . If we expand  $f(x_n, r)$  about  $(x(i), r_0)$ , we have to first-order:

$$x_{n+1} - x(i+1) = \frac{\partial f(x, r)}{\partial x} [x_n - x(i)] + \frac{\partial f(x, r)}{\partial r} \Delta r = 0. \quad (6.29)$$

The partial derivatives in (6.29) are evaluated at  $x = x(i)$  and  $r = r_0$ . The result can be expressed as

$$4r_0[1 - 2x(i)][x_n - x(i)] + 4x(i)[1 - x(i)]\Delta r = 0. \quad (6.30)$$

The solution of (6.30) for  $\Delta r$  can be written as

$$\Delta r = -r_0 \frac{[1 - 2x(i)][x_n - x(i)]}{x(i)[1 - x(i)]}. \quad (6.31)$$

The procedure is to iterate the logistic map at  $r = r_0$  until  $x_n$  is sufficiently close to a  $x(i)$ . The nature of chaotic systems is that the trajectory is guaranteed to come close to the desired unstable trajectory eventually. Then we use (6.31) to change the value of  $r$  so that the next iteration is closer to  $x(i+1)$ . We summarize the algorithm for controlling chaos as follows:

1. Find the unstable periodic trajectory  $x(1), x(2) \dots x(p)$ , for the desired value of  $r_0$ .
2. Iterate the map with  $r = r_0$  until  $x_n$  is within  $\epsilon$  of  $x(i)$ . Then use (6.31) to determine  $r$ .
3. To turn off the control, set  $r = r_0$ .

**Problem 6.14.** Controlling chaos

- a. Write a program that allows the user to turn the control on and off. The trajectory can be seen by plotting  $x_n$  versus  $n$ . The program should incorporate as input the desired unstable periodic trajectory  $x(i)$ , the period  $p$ , the value of  $r_0$ , and the parameter  $\epsilon$ .
- b. Test your program with  $r_0 = 0.95$  and the periods  $p = 1, 5$ , and  $13$ . Use  $\epsilon = 0.02$ .
- c. Modify your program so that the values of  $r$  are shown as well as the values of  $x_n$ . How does  $r$  change if we vary  $\epsilon$ ? Try  $\epsilon = 0.05, 0.01$ , and  $0.005$ .
- d. Add a method to compute  $n_\epsilon$ , the number of iterations necessary for the trajectory  $x_n$  to be within  $\epsilon$  of  $x(1)$  when the control is on. Find  $\langle n_\epsilon \rangle$ , the average value of  $n_\epsilon$ , by starting with 100 random values of  $x_0$ . Compute  $\langle n_\epsilon \rangle$  as a function of  $\epsilon$  for  $\delta = 0.05, 0.005, 0.0005$ , and  $0.00005$ . What is the functional dependence of  $\langle n_\epsilon \rangle$  on  $\epsilon$ ?

## 6.7 \*Higher-Dimensional Models

So far we have discussed the logistic map as a mathematical model that has some remarkable properties and produces some interesting computer graphics. In this section we discuss some two- and three-dimensional systems that also might seem to have little to do with realistic physical systems. However, as we will see in Sections 6.8 and 6.9, similar behavior is found in realistic physical systems under the appropriate conditions.

We begin with a two-dimensional map and consider the sequence of points  $(x_n, y_n)$  generated by

$$x_{n+1} = y_n + 1 - ax_n^2 \quad (6.32a)$$

$$y_{n+1} = bx_n. \quad (6.32b)$$

The map (6.32) was proposed by Hénon who was motivated by the relevance of this dynamical system to the behavior of asteroids and satellites.

**Problem 6.15.** The Hénon map

- a. Write a program to iterate (6.32) for  $a = 1.4$  and  $b = 0.3$  and plot  $10^4$  iterations starting from  $x_0 = 0, y_0 = 0$ . Make sure you compute the new value of  $y$  using the old value of  $x$  and not the new value of  $x$ . Do not plot the initial transient. Look at the trajectory in the region defined by  $|x| \leq 1.5$  and  $|y| \leq 0.45$ . Make a similar plot beginning from the second initial condition,  $x_0 = 0.63135448, y_0 = 0.18940634$ . Compare the shape of the two plots. Is the shape of the two curves independent of the initial conditions?

- b. Increase the scale of your plot so that all points in the region  $0.50 \leq x \leq 0.75$  and  $0.15 \leq y \leq 0.21$  are shown. Begin from the second initial condition and increase the number of computed points to  $10^5$ . Then make another plot showing all points in the region  $0.62 \leq x \leq 0.64$  and  $0.185 \leq y \leq 0.191$ . If patience permits, make an additional enlargement and plot all points within the box defined by  $0.6305 \leq x \leq 0.6325$  and  $0.1889 \leq y \leq 0.1895$ . (You will have to increase the number of computed points to order  $10^6$ .) What is the structure of the curves within each box? Does the attractor appear to have a similar structure on smaller and smaller length scales? Is there a region in the plane from which the points cannot escape? The region of points that do not escape is the basin of the Hénon attractor. The attractor itself is the set of points to which all points in the basin are attracted. That is, two trajectories that begin from different conditions will soon lie on the attractor.
- c. Determine if the system is chaotic, that is, sensitive to initial conditions. Start two points very close to each other and watch their trajectories for a fixed time. Choose different colors for the two trajectories.
- d. It is straightforward in principle to extend the method for computing the Lyapunov exponent that we used for a one-dimensional map to higher-dimensional maps. The idea is to linearize the difference (or differential) equations and replace  $dx_n$  by the corresponding vector quantity  $d\mathbf{x}_n$ . This generalization yields the Lyapunov exponent corresponding to the divergence along the fastest growing direction. If a system has  $f$  degrees of freedom, it has a set of  $f$  Lyapunov exponents. A method for computing all  $f$  exponents is discussed in Project 6.24.

One of the earliest indications of chaotic behavior was in an atmospheric model developed by Lorenz. His goal was to describe the motion of a fluid layer that is heated from below. The result is convective rolls, where the warm fluid at the bottom rises, cools off at the top, and then falls down later. Lorenz simplified the description by restricting the motion to two spatial dimensions. This situation has been modeled experimentally in the laboratory and is known as a Rayleigh-Benard cell. The equations that Lorenz obtained are

$$\frac{dx}{dt} = -\sigma x + \sigma y \quad (6.33a)$$

$$\frac{dy}{dt} = -xz + rx - y \quad (6.33b)$$

$$\frac{dz}{dt} = xy - bz, \quad (6.33c)$$

where  $x$  is a measure of the fluid flow velocity circulating around the cell,  $y$  is a measure of the temperature difference between the rising and falling fluid regions, and  $z$  is a measure of the difference in the temperature profile between the bottom and the top from the normal equilibrium temperature profile. The dimensionless parameters  $\sigma$ ,  $r$ , and  $b$  are determined by various fluid properties, the size of the Rayleigh-Benard cell, and the temperature difference in the cell. Note that the variables  $x$ ,  $y$ , and  $z$  have nothing to do with the spatial coordinates, but are measures of the state of the system. Although it is not expected that you will understand the relation of the Lorenz equations to convection, we have included these equations here to reinforce the idea that simple sets of equations can exhibit chaotic behavior.

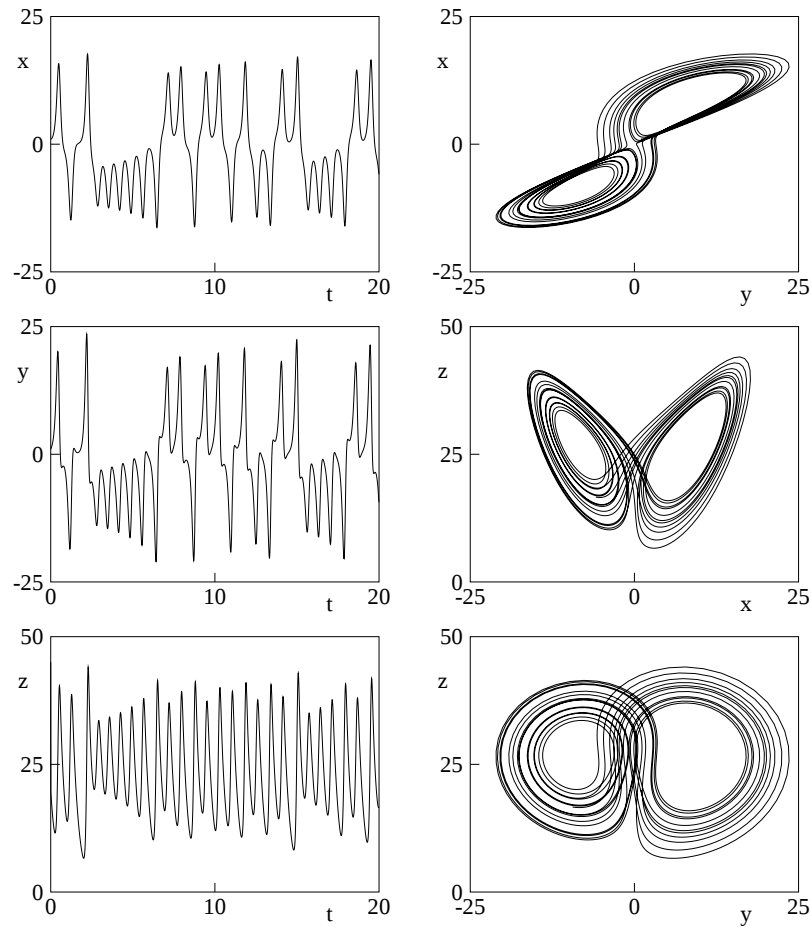


Figure 6.10: A trajectory of the Lorenz model with  $\sigma = 10$ ,  $b = 8/3$ , and  $r = 28$  and the initial condition  $x_0 = 1, y_0 = 1, z_0 = 20$ . A time interval of  $t = 20$  is shown with points plotted at intervals of 0.01. The fourth-order Runge-Kutta algorithm was used with  $\Delta t = 0.0025$ .

To make three-dimensional plots, we use the `EJSFrame` class; the only argument of its constructor is the title for the plot. The following code sets up a plot where the class `Lorenz` contains the code for drawing the attractor.

```
EJSFrame frame = new EJSFrame("Lorenz Attractor");
Lorenz lorenz = new Lorenz();
frame.setPreferredMinMax ( -15.0, 15.0, -15.0, 15.0, 0.0, 50.0);
frame.setDecorationType(DrawingPanel3D.DECORATION_AXES);
frame.addDrawable(lorenz);
```

`Lorenz` extends `GroupDrawable` and must import `org.opensourcephysics.displayejs.*`. Two useful drawing objects are

```
InteractiveParticle ball = new
InteractiveParticle ( InteractiveParticle .ELLIPSE);
InteractiveTrace trace =
new InteractiveTrace();
```

The properties of these objects are determined by

```
ball.setSizeXYZ(1, 1, 1); // sets size of ball in world coordinates
ball.getStyle().setFillPattern(java.awt.Color.RED); // sets color
add(trace); // adds trace to Lorenz
add(ball); // adds ball to Lorenz
```

To plot each part of the trace, use the method `trace.addpoint(x,y,z)` and to plot the current position, use `ball.setXYZ(x,y,z)`. The user can then rotate the three-dimensional plot using the mouse.

**Problem 6.16.** The Lorenz model

- Use a Runge-Kutta algorithm (see Appendix 3A) to obtain a numerical solution of the Lorenz equations (6.33). Generate three-dimensional plots using `EJSFrame`. Explore the basin of the attractor with  $\sigma = 10$ ,  $b = 8/3$ , and  $r = 28$ .
- Determine qualitatively the sensitivity to initial conditions. Start two points very close to each other and watch their trajectories for approximately  $10^4$  time steps.
- Let  $z_m$  denote the value of  $z$  where  $z$  is a relative maximum for the  $m$ th time. You can determine the value of  $z_m$  by finding the average of the two values of  $z$  when the right-hand side of (6.33) changes sign. Plot  $z_{m+1}$  versus  $z_m$  and describe what you find. This procedure is one way that a continuous system can be mapped onto a discrete map. What is the slope of the  $z_{m+1}$  versus  $z_m$  curve? Is its magnitude always greater than unity? If so, then this behavior is an indication of chaos. Why?

The application of the Lorenz equations to weather prediction has led to a popular metaphor known as the *butterfly effect*. This metaphor is made even more meaningful by inspection of Figure 6.10. The “butterfly effect” is often ascribed to Lorenz. In a 1963 paper he remarked that:

“One meteorologist remarked that if the theory were correct, one flap of a seagull’s wings would be enough to alter the course of the weather forever.”

By 1972, the sea gull had evolved into the more poetic butterfly and the title of his talk was “Predictability: Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?”

## 6.8 Forced Damped Pendulum

We now consider the dynamics of nonlinear mechanical systems described by classical mechanics. The general problem in classical mechanics is the determination of the positions and velocities of a system of particles subjected to certain forces. For example, we considered in Chapter 5 the celestial two-body problem and were able to predict the motion at any time. We will find that we

cannot make long-time predictions for the trajectories of nonlinear classical systems when these systems exhibit chaos.

A familiar example of a nonlinear mechanical system is the simple pendulum (see Chapter 3). To make its dynamics more interesting, we assume that there is a linear damping term present and that the pivot is forced to move vertically up and down. Newton's second law for this system is (cf. McLaughlin or Percival and Richards)

$$\frac{d^2\theta}{dt^2} = -\gamma \frac{d\theta}{dt} - [\omega_0^2 + 2A \cos \omega t] \sin \theta, \quad (6.34)$$

where  $\theta$  is the angle the pendulum makes with the vertical axis,  $\gamma$  is the damping coefficient,  $\omega_0^2 = g/L$  is the natural frequency of the pendulum, and  $\omega$  and  $A$  are the frequency and amplitude of the external force. Note that the effect of the vertical acceleration of the pivot is equivalent to a time-dependent gravitational field.

How do we expect the driven, damped simple pendulum to behave? Because there is damping present, we expect that if there is no external force, the pendulum would come to rest. That is,  $(x = 0, v = 0)$  is a stable attractor. As  $A$  is increased from zero, this attractor remains stable for sufficiently small  $A$ . At a value of  $A$  equal to  $A_c$ , this attractor becomes unstable. How does the driven nonlinear oscillator behave as we increase the amplitude  $A$ ?

Because we are mainly interested in the (stable and unstable) fixed points of the motion, it is convenient to analyze the motion by plotting a point in phase space after every cycle of the external force. Such a phase space plot is called a *Poincaré map*. Hence, we will plot  $d\theta/dt$  versus  $\theta$  for values of  $t$  equal to  $nT$  for  $n$  equal to  $1, 2, 3, \dots$ . If the system has a period  $T$ , then the Poincaré map consists of a single point. If the period of the system is  $nT$ , there will be  $n$  points.

`PoincareApp` uses the fourth-order Runge-Kutta algorithm to compute  $\theta(t)$  and the angular velocity  $d\theta(t)/dt$  for the pendulum described by (6.34). This equation is modeled in the `DampedDrivenPendulum` class, but is not shown here because it is similar to other ODE implementations. A phase diagram for  $d\theta(t)/dt$  versus  $\theta(t)$  is shown in one frame. In the other frame the Poincaré map is represented by drawing a small box at the point  $(\theta, d\theta/dt)$  at time  $t = nT$ . If the system has period 1, that is, if the same values of  $(\theta, d\theta/dt)$  are drawn at  $t = nT$ , we would see only one box; otherwise we would see several boxes. Because the first few values of  $(\theta, d\theta/dt)$  show the transient behavior, it is desirable to clear the display and draw a new Poincaré map without changing  $A$ ,  $\theta$ , or  $d\theta/dt$ .

Listing 6.5: `PoincareApp` plots a phase diagram and a Poincaré map for the damped driven pendulum.

```
package org.opensourcephysics.sip.ch06;
import org.opensourcephysics.controls.AbstractSimulation;
import org.opensourcephysics.controls.SimulationControl;
import org.opensourcephysics.frames.PlotFrame;
import org.opensourcephysics.numerics.RK4;

public class PoincareApp extends AbstractSimulation {
    final static double PI = Math.PI; // defined for brevity
    PlotFrame phaseSpace = new PlotFrame("theta", "ang vel", "Phase space plot");
    PlotFrame poincare = new PlotFrame("theta", "ang vel", "Poincare plot");
```

```

int nstep = 100;                // # iterations between Poincare plot
DampedDrivenPendulum pendulum = new DampedDrivenPendulum();
RK4 odeMethod = new RK4(pendulum);
public PoincareApp() {
    // angular frequency of external force equals two and hence period of external force equals pi
    odeMethod.setStepSize(PI/nstep); // dt = PI/nsteps
    phaseSpace.setMarkerShape(0, 6); // second argument indicates a pixel
    poincare.setMarkerSize(0, 2);    // smaller size gives better resolution
    poincare.setMarkerColor(0, java.awt.Color.RED);
    phaseSpace.setMessage("t="+0);
}

public void reset() {
    control.setValue("theta", 0.2);
    control.setValue("angular velocity", 0.6);
    control.setValue("gamma", 0.2); // damping constant
    control.setValue("A", 0.85);    // amplitude
}

public void doStep() {
    double state[] = pendulum.getState();
    for(int istep = 0; istep < nstep; istep++) {
        odeMethod.step();
        if(state[0] > PI) {
            state[0] = state[0] - 2.0 * PI;
        } else if (state[0] < -PI) {
            state[0] = state[0] + 2 * PI;
        }
        phaseSpace.append(0, state[0], state[1]);
    }
    poincare.append(0, state[0], state[1]);
    phaseSpace.setMessage("t=" + decimalFormat.format(state[2]));
    poincare.setMessage("t=" + decimalFormat.format(state[2]));
    if(phaseSpace.isShowing()) {
        phaseSpace.render();
    }
    if(poincare.isShowing()) {
        poincare.render();
    }
}

public void initialize () {
    double theta = control.getDouble("theta"); // initial angle
    double omega = control.getDouble("angular velocity"); // initial angular velocity
    pendulum.gamma = control.getDouble("gamma"); // damping constant
    pendulum.A = control.getDouble("A"); // amplitude of external force
    pendulum.initializeState(new double[]{theta, omega, 0});
    clear ();
}

```

```

    public void clear() {
        phaseSpace.clearData();
        poincare.clearData();
        phaseSpace.render();
        poincare.render();
    }

    public static void main(String[] args) {
        SimulationControl control = SimulationControl.createApp(new PoincareApp());
        control.addButton("clear", "Clear");
    }
}

```

**Problem 6.17.** Dynamics of a driven, damped simple pendulum

- a. Use `PoincareApp` to simulate the driven, damped simple pendulum. In the program  $\omega = 2$  so that the period  $T$  of the external force equals  $\pi$ . The program also assumes that  $\omega_0 = 1$ . Use  $\gamma = 0.2$  and  $A = 0.85$  and compute the phase space trajectory. After the initial transient, how many points do you see in the Poincaré plot? What is the period of the pendulum? Vary the initial values of  $\theta$  and  $d\theta/dt$ . Is the attractor independent of the initial conditions? Remember to ignore the initial transient behavior.
- b. Modify `PoincareApp` so that it plots  $\theta$  and  $d\theta/dt$  as a function of  $t$ . Describe the qualitative relation between the Poincaré plot, the phase space plot, and the  $t$  dependence of  $\theta$  and  $d\theta/dt$ .
- c. The amplitude  $A$  plays the role of the control parameter for the dynamics of the system. Use the behavior of the Poincaré plot to find the value  $A = A_c$  at which the  $(0, 0)$  attractor becomes unstable. Start with  $A = 0.1$  and continue increasing  $A$  until the  $(0, 0)$  attractor becomes unstable.
- d. Find the period for  $A = 0.1, 0.25, 0.5, 0.7, 0.75, 0.85, 0.95, 1.00, 1.02, 1.031, 1.033, 1.036$ , and  $1.05$ . Note that for small  $A$ , the period of the oscillator is twice that of the external force. The steady state period is  $2\pi$  for  $A_c < A < 0.71$ ,  $\pi$  for  $0.72 < A < 0.79$ , and then  $2\pi$  again.
- e. The first period-doubling occurs for  $A \approx 0.79$ . Find the approximate values of  $A$  for further period-doubling and use these values of  $A$  to compute the exponent  $\delta$  defined by (6.10). Compare your result for  $\delta$  with the result found for the one-dimensional logistic map. Are your results consistent with those that you found for the logistic map? An analysis of this system can be found in the article by McLaughlin.
- f. Sometimes a trajectory does not approach a steady state even after a very long time, but a slight perturbation causes the trajectory to move quickly onto a steady state attractor. Consider  $A = 0.62$  and the initial condition  $(\theta = 0.3, d\theta/dt = 0.3)$ . Describe the behavior of the trajectory in phase space. During the simulation, change  $\theta$  by  $0.1$ . Does the trajectory move onto a steady state trajectory? Do similar simulations for other values of  $A$  and other initial conditions.
- g. Repeat the calculations of parts (b)–(d) for  $\gamma = 0.05$ . What can you conclude about the effect of damping?

- h. Replace the fourth-order Runge-Kutta algorithm by the lower-order Euler-Richardson algorithm. Which algorithm gives the better trade-off between accuracy and speed?

**Problem 6.18.** Basin of an attractor

- a. For  $\gamma = 0.2$  and  $A > 0.79$  the pendulum rotates clockwise or counterclockwise in the steady state. Each of these two rotations is an attractor. The set of initial conditions that lead to a particular attractor is called the basin of the attractor. Modify `PoincareApp` so that the program draws the basin of the attractor with  $d\theta/dt > 0$ . For example, your program might simulate the motion for about 20 periods and then determine the sign of  $d\theta/dt$ . If  $d\theta/dt > 0$  in the steady state, then the program plots a point in phase space at the coordinates of the initial condition. The program repeats this process for many initial conditions. Describe the basin of attraction for  $A = 0.85$  and increments of the initial values of  $\theta$  and  $d\theta/dt$  equal to  $\pi/10$ .
- b. Repeat part (a) using increments of the initial values of  $\theta$  and  $d\theta/dt$  equal to  $\pi/20$  or as small as possible given your computer resources. Does the boundary of the basin of attraction appear smooth or rough? Is the basin of the attractor a single region or is it disconnected into more than one piece?
- c. Repeat parts (a) and (b) for other values of  $A$ , including values near the onset of chaos and in the chaotic regime. Is there a qualitative difference between the basins of periodic and chaotic attractors? For example, can you always distinguish the boundaries of the basin?

## 6.9 \*Hamiltonian Chaos

Hamiltonian systems are a very important class of dynamical systems. The most familiar are mechanical systems without friction, and the most important of these is the solar system. The linear harmonic oscillator and the simple pendulum that we considered in Chapter 3 are two simple examples. Many other systems can be included in the Hamiltonian framework, for example, the motion of charged particles in electric and magnetic fields, and ray optics. The Hamiltonian dynamics of charged particles is particularly relevant to confinement issues in particle accelerators, storage rings, and plasmas. In each case a function of all the coordinates and momenta called the Hamiltonian is formed. For many mechanical systems this function can be identified with the total energy. The Hamiltonian for a particle in a potential  $V(x, y, z)$  is

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z). \quad (6.35)$$

Typically we write (6.35) using the notation

$$H = \sum_i \frac{p_i^2}{2m} + V(\{q_i\}), \quad (6.36)$$

where  $p_1 \equiv p_x$ ,  $q_1 \equiv x$ , etc. This notation emphasizes that the  $p_i$  and the  $q_i$  are generalized coordinates. For example, in some systems  $p$  can represent the angular momentum and  $q$  can

represent an angle. For a system of  $N$  particles in three dimensions, the sum in (6.36) runs from 1 to  $3N$ , where  $3N$  is the number of degrees of freedom.

The methods for constructing the generalized momenta and the Hamiltonian are described in standard classical mechanics texts. The time dependence of the generalized momenta and coordinates is given by

$$\dot{p}_i \equiv \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (6.37a)$$

$$\dot{q}_i \equiv \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad (6.37b)$$

Check that (6.37) leads to the usual form of Newton's second law by considering the simple example of a single particle in a potential.

As we found in Chapter 3, an important property of conservative systems is preservation of areas in phase space. Consider a set of initial conditions of a dynamical system that form a closed surface in phase space. For example, if phase space is two-dimensional, this surface would be a one-dimensional loop. As time evolves, this surface in phase space will typically change its shape. For Hamiltonian systems the volume enclosed by this surface remains constant in time. For dissipative systems this volume will decrease, and hence dissipative systems are not described by a Hamiltonian. One consequence of the constant phase space volume is that Hamiltonian systems do not have phase space attractors.

In general, the motion of Hamiltonian systems is very complex. In some systems the motion is regular, and there is a constant of the motion (a quantity that does not change with time) for each degree of freedom. Such a system is said to be *integrable*. For time independent systems an obvious constant of the motion is the total energy. The total momentum and angular momentum are other examples. There may be others as well. If there are more degrees of freedom than constants of the motion, then the system can be chaotic. When the number of degrees of freedom becomes large, the possibility of chaotic behavior becomes more likely. An important example that we will consider in Chapter 8 is a collection of interacting particles. Their chaotic motion is essential for the system to be described by the methods of statistical mechanics.

For regular motion the change in shape of a closed surface in phase space would be rather uninteresting. For chaotic motion, nearby trajectories must exponentially diverge from each other, but are confined to a finite region of phase space. Hence, there will be local stretching of the surface accompanied by repeated folding to ensure confinement. There is another class of systems whose behavior is in between, that is, the system behaves regularly for some initial conditions, and chaotically for others. We will study these *mixed* systems in this section.

Consider the Hamiltonian for a system of  $N$  particles. If the system is integrable, there are  $3N$  constants of the motion. It is natural to identify the generalized momenta with these constants. The coordinates that are associated with each of these constants will vary linearly with time. If the system is confined in phase space, then the coordinates must be periodic. If we have just one coordinate, we can think of the motion as being a point moving on a circle in phase space. In two dimensions the motion is a point moving in two circles at once, that is, a point moving on the surface of a torus. In three dimensions we can imagine a generalized torus with three circles, and so on. If the period of motion along each circle is a rational fraction of the period of all the other

circles, then the torus is called a resonant torus, and the motion in phase space is periodic. If the periods are not rational fractions of each other, then the torus is called nonresonant.

If we take an integrable Hamiltonian and change it slightly, what happens to these tori? A partial answer is given by a theorem due to Kolmogorov, Arnold, and Moser (KAM), which states that, under certain circumstances, these tori will remain. When the perturbation of the Hamiltonian becomes large enough, these KAM tori are destroyed.

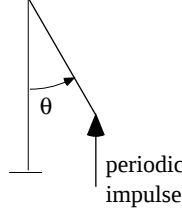


Figure 6.11: Model of a kicked rotor consisting of a rigid rod with moment of inertia  $I$ . Gravity and friction at the pivot is ignored. The motion of the rotor is given by the standard map in (6.39).

To understand the basic ideas associated with mixed systems, we consider a simple model known as the *standard map*. Consider the rotor shown in Figure 6.11. The rod has a moment of inertia  $I$  and length  $L$  and is fastened at one end to a frictionless pivot. The other end is subjected to a vertical periodic impulsive force of strength  $k/L$  applied at time  $t = 0, \tau, 2\tau, \dots$ . Gravity is ignored. The motion of the rotor can be described by the angle  $\theta$  and the corresponding angular momentum  $p_\theta$ . The Hamiltonian for this system can be written as

$$H(\theta, p_\theta, t) = \frac{p_\theta^2}{2I} + k \cos \theta \sum_n \delta(t - n\tau). \quad (6.38)$$

The term  $\delta(t - n\tau)$  is zero everywhere except at  $t = n\tau$ ; its integral over time is unity if  $t = n\tau$  is within the limits of integration. If we use (6.37) and (6.38), it is easy to show that the corresponding equations of motion are given by

$$\frac{dp_\theta}{dt} = k \sin \theta \sum_n \delta(t - n\tau) \quad (6.39a)$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{I}. \quad (6.39b)$$

From (6.39) we see that  $p_\theta$  is constant between kicks (remember that gravity is assumed to be absent), but changes discontinuously at each kick. The angle  $\theta$  varies linearly with  $t$  between kicks and is continuous at each kick.

It is convenient to know the values of  $\theta$  and  $p_\theta$  at times just after the kick. We let  $\theta_n$  and  $p_n$  be the values of  $\theta(t)$  and  $p_\theta(t)$  at times  $t = n\tau + 0^+$ , where  $0^+$  is an infinitesimally small positive number. If we integrate (6.39a) from  $t = (n+1)\tau - 0^+$  to  $t = (n+1)\tau + 0^+$ , we obtain

$$p_{n+1} - p_n = k \sin \theta_{n+1}. \quad (6.40a)$$

(Remember that  $p$  is constant between kicks and the delta function contributes to the integral only when  $t = (n+1)\tau$ .) From (6.39b) we have

$$\theta_{n+1} - \theta_n = (\tau/I)p_n. \quad (6.40b)$$

If we choose units such that  $\tau/I = 1$ , we obtain the standard map

$$\theta_{n+1} = (\theta_n + p_n) \text{ modulo } 2\pi, \quad (6.41a)$$

$$p_{n+1} = p_n + k \sin \theta_{n+1}. \quad (\text{standard map}) \quad (6.41b)$$

We have added the requirement in (6.41a) that the value of the angle  $\theta$  is restricted to be between zero and  $2\pi$ .

Before we iterate (6.41), let us check that (6.41) represents a Hamiltonian system, that is, the area in  $q$ - $p$  space is constant as  $n$  increases. (Here  $q$  corresponds to  $\theta$ .) Suppose we start with a rectangle of points of length  $dq_n$  and  $dp_n$ . After one iteration, this rectangle will be deformed into a parallelogram of sides  $dq_{n+1}$  and  $dp_{n+1}$ . From (6.41) we have

$$dq_{n+1} = dq_n + dp_n \quad (6.42a)$$

$$dp_{n+1} = dp_n + k \cos q_{n+1} dq_{n+1}. \quad (6.42b)$$

If we substitute (6.42a) in (6.42b), we obtain

$$dp_{n+1} = (1 + k \cos q_{n+1}) dp_n + k \cos q_{n+1} dq_n. \quad (6.43)$$

To find the area of a parallelogram, we take the magnitude of the cross product of the vectors  $d\mathbf{q}_{n+1} = (dq_n, dp_n)$  and  $d\mathbf{p}_{n+1} = (1 + k \cos q_n dq_n, k \cos q_n dp_n)$ . The result is  $dq_n dp_n$ , and hence the area in phase space has not changed. We say that the standard map is an example of an *area-preserving map*.

The qualitative properties of the standard map are explored in Problem 6.19. A summary of its properties follows. For  $k = 0$ , the rod rotates with a fixed angular velocity determined by the momentum  $p_n = p_0 = \text{constant}$ . If  $p_0$  is a rational number times  $2\pi$ , then the trajectory in phase space consists of a sequence of isolated points lying on a horizontal line (resonant tori). Can you see why? If  $p_0$  is not a rational number times  $2\pi$  or if your computer does not have sufficient precision, then after a long enough time, the trajectory will consist of a horizontal line in phase space. As we increase  $k$ , these horizontal lines are deformed into curves that run from  $q = 0$  to  $q = 2\pi$ , and the isolated points of the resonant tori are converted into closed loops. For some initial conditions, the trajectories will become chaotic after the map has been iterated long enough so that the transient behavior has ended.

**Problem 6.19.** The standard map

- Write a program to iterate the standard map and plot its trajectory in phase space. Design the program so that more than one trajectory for the same value of the parameter  $k$  can be shown at the same time (using different colors). Choose a set of initial conditions that form a rectangle (see Problem 4.10). Does the shape of this area change with time? What happens to the total area?
- Begin with  $k = 0$  and choose an initial value of  $p$  that is a rational number times  $2\pi$ . What types of trajectories do you obtain? If you obtain trajectories consisting of isolated points, do these points appear to shift due to numerical roundoff errors? How can you tell? What happens if  $p_0$  is an irrational number times  $2\pi$ ? Remember that a computer can only approximate an irrational number.

- c. Consider  $k = 0.2$  and explore the nature of the phase space trajectories. What structures appear that do not appear for  $k = 0$ ? Discuss the motion of the rod corresponding to some of the typical trajectories that you find.
- d. Increase  $k$  until you first find some chaotic trajectories. How can you tell that they are chaotic? Do these chaotic trajectories fill all of phase space? If there is one trajectory that is chaotic at a particular value of  $k$ , are all trajectories chaotic? What is the approximate value for  $k_c$  above which chaotic trajectories appear?

We now discuss a discrete map that models the rings of the planet Saturn (see Fröyland). The assumption is that the rings of Saturn are due to perturbations produced by Mimas, one of Saturn's moons, which is a distance of  $\sigma = 185.7 \times 10^3$  km from Saturn. There are two important forces acting on objects near Saturn. The force due to Saturn can be incorporated as follows. We know that each time Mimas completes an orbit, it traverses a total angle of  $2\pi$ . Hence, the angle  $\theta$  of any other moon of Saturn relative to Mimas can be expressed as

$$\theta_{n+1} = \theta_n + 2\pi \frac{\sigma^{3/2}}{r_n^{3/2}}, \quad (6.44)$$

where  $r_n$  is the radius of the orbit after  $n$  revolutions. The other important force is due to Mimas and causes the radial distance  $r_n$  to change. A discrete approximation to the radial acceleration  $dv_r/dt$  is

$$\begin{aligned} \frac{\Delta v_r}{\Delta t} &\approx \frac{v_r(t + \Delta t) - v_r(t)}{\Delta t} \\ &\approx \frac{r(t + \Delta t) - r(t)}{(\Delta t)^2} - \frac{r(t) - r(t - \Delta t)}{(\Delta t)^2} \\ &= \frac{r(t + \Delta t) - 2r(t) + r(t - \Delta t)}{(\Delta t)^2}. \end{aligned} \quad (6.45)$$

The acceleration equals the radial force due to Mimas. If we average over a complete period, then a reasonable approximation for the change in  $r_n$  due to Mimas is

$$r_{n+1} - 2r_n + r_{n-1} = f(r_n, \theta_n), \quad (6.46)$$

where  $f(r_n, \theta_n)$  is the radial force.

In general, the form of  $f(r_n, \theta_n)$  is very complicated. We make a major simplifying assumption and take  $f$  to be proportional to  $-(r_n - \sigma)^{-2}$  and to be periodic in  $\theta_n$ . For simplicity, we express this periodicity in the simplest possible way, that is, as  $\cos \theta_n$ . We also want the map to be area conserving. These considerations lead to the following two-dimensional map:

$$\theta_{n+1} = \theta_n + 2\pi \frac{\sigma^{3/2}}{r_n^{3/2}} \quad (6.47a)$$

$$r_{n+1} = 2r_n - r_{n-1} - a \frac{\cos \theta_n}{(r_n - \sigma)^2}. \quad (6.47b)$$

The constant  $a$  for Saturn's rings is approximately  $2 \times 10^{12} \text{ km}^3$ . We can show, using a similar technique as before, that the volume in  $(r, \theta)$  space is preserved, and hence (6.47) is a Hamiltonian map.

The purpose of the above discussion was only to motivate and not to derive the form of the map (6.47). In Problem 6.20 we investigate how the map (6.47) yields the qualitative structure of Saturn's rings. In particular, what happens to the values of  $r_n$  if the period of a moon is related to the period of Mimas by the ratio of two integers?

**Problem 6.20.** A simple model of the rings of Saturn

- Write a program to implement the map (6.47). Be sure to save the last two values of  $r$  so that the values of  $r_n$  are updated correctly. The radius of Saturn is  $60.4 \times 10^3$  km. Express all lengths in units of  $10^3$  km. In these units  $a = 2000$ . Plot the points  $(r_n \cos \theta_n, r_n \sin \theta_n)$ . Choose initial values for  $r$  between the radius of Saturn and  $\sigma$ , the distance of Mimas from Saturn, and find the bands of  $r_n$  values where stable trajectories are found.
- What is the effect of changing the value of  $a$ ? Try  $a = 200$  and  $a = 20000$  and compare your results with part (a).
- Vary the force function. Replace  $\cos \theta$  by other trigonometric functions. How do your results change? If the changes are small, does that give you some confidence that the model has something to do with Saturn's rings?

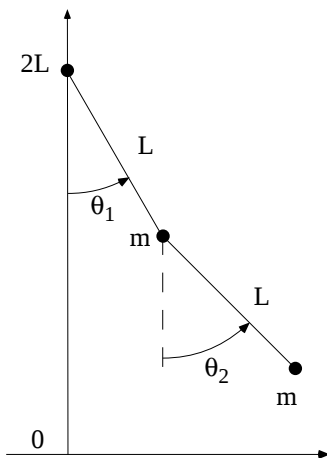


Figure 6.12: The double pendulum.

A more realistic dynamical system is the double pendulum, a system that can be demonstrated in the laboratory. This system consists of two equal point masses  $m$ , with one suspended from a fixed support by a rigid weightless rod of length  $L$ , and the other suspended from the first by a similar rod (see Figure 6.12). Because there is no friction, this system is clearly an example of a Hamiltonian system. The four rectangular coordinates  $x_1, y_1, x_2$ , and  $y_2$  of the two masses can be

expressed in terms of two generalized coordinates  $\theta_1, \theta_2$ :

$$x_1 = L \sin \theta_1 \quad (6.48a)$$

$$y_1 = 2L - L \cos \theta_1 \quad (6.48b)$$

$$x_2 = L \sin \theta_1 + L \sin \theta_2 \quad (6.48c)$$

$$y_2 = 2L - L \cos \theta_1 - L \cos \theta_2. \quad (6.48d)$$

The kinetic energy is given by

$$K = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) = \frac{1}{2}mL^2[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)], \quad (6.49)$$

and the potential energy is given by

$$U = mgL(3 - 2 \cos \theta_1 - \cos \theta_2). \quad (6.50)$$

To use Hamilton's equations of motion (6.37), we need to express the sum of the kinetic energy and potential energy in terms of the generalized momenta and coordinates. In rectangular coordinates we know that the momenta are equal to  $p_i = \partial K / \partial \dot{q}_i$ , where for example,  $q_i = x_1$  and  $p_i$  is the  $x$ -component of  $m\mathbf{v}_1$ . This prescription works for generalized momenta as well, and the generalized momentum corresponding to  $\theta_1$  is given by  $p_1 = \partial K / \partial \dot{\theta}_1$ . If we calculate the appropriate derivatives, we can show that the generalized momenta can be written as

$$p_1 = mL^2[2\dot{\theta}_1 + \dot{\theta}_2 \cos(\theta_1 - \theta_2)] \quad (6.51a)$$

$$p_2 = mL^2[\dot{\theta}_2 + \dot{\theta}_1 \cos(\theta_1 - \theta_2)]. \quad (6.51b)$$

The Hamiltonian or total energy becomes

$$H = \frac{1}{2mL^2} \frac{p_1^2 + 2p_2^2 - 2p_1p_2 \cos(q_1 - q_2)}{1 + \sin^2(q_1 - q_2)} + mgL(3 - 2 \cos q_1 - \cos q_2), \quad (6.52)$$

where  $q_1 = \theta_1$  and  $q_2 = \theta_2$ . The equations of motion can be found by using (6.52) and (6.37).

Figure 6.13 shows a Poincaré map for the double pendulum. The coordinate  $p_1$  is plotted versus  $q_1$  for the same total energy  $E = 15$ , but for two different initial conditions. The map includes the points in the trajectory for which  $q_2 = 0$  and  $p_2 > 0$ . Note the resemblance between Figure 6.13 and plots for the standard map above the critical value of  $k$ , that is, there is a regular trajectory and a chaotic trajectory for the same parameters, but different initial conditions.

### Problem 6.21. Double pendulum

- Use either the fourth-order Runge-Kutta algorithm (with  $\Delta t = 0.003$ ) or the second-order Euler-Richardson algorithm (with  $\Delta t = 0.001$ ) to simulate the double pendulum. Choose  $m = 1$ ,  $L = 1$ , and  $g = 9.8$ . The input parameter is the total energy  $E$ . The initial values of  $q_1$  and  $q_2$  can be either chosen randomly within the interval  $-\pi < q_i < \pi$  or by the user. Then set the initial  $p_1 = 0$ , and solve for  $p_2$  using (6.52) with  $H = E$ . First explore the pendulum's behavior by plotting the generalized coordinates and momenta as a function of time in four windows.

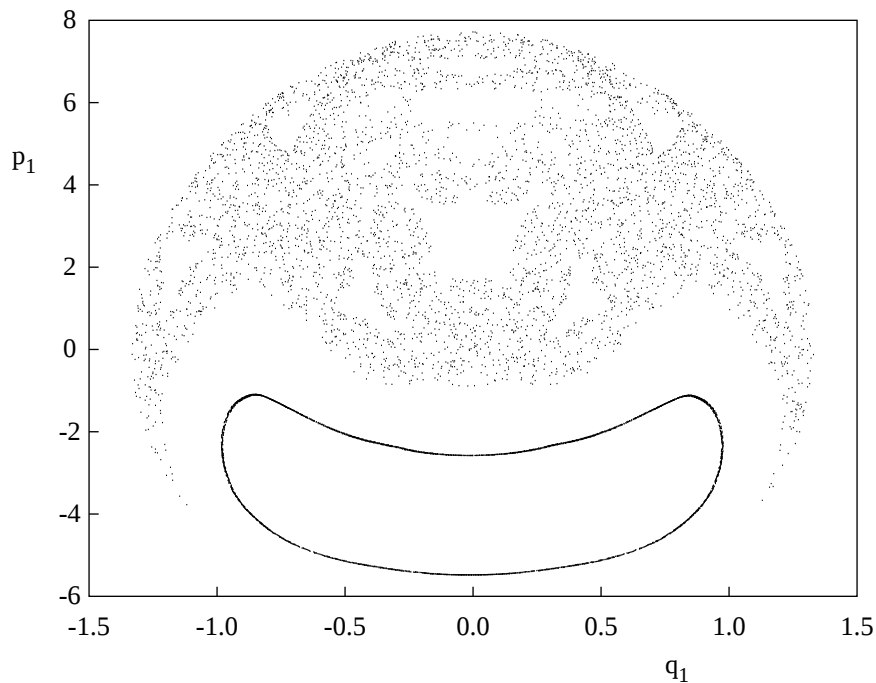


Figure 6.13: Poincaré plot for the double pendulum with  $p_1$  plotted versus  $q_1$  at  $q_2 = 0$  and  $p_2 > 0$ . Two sets of initial conditions,  $(q_1, q_2, p_1) = (0, 0, 0)$  and  $(1.1, 0, 0)$  respectively, were used to create the plot. The initial value of the coordinate  $p_2$  is found from (6.52) by requiring that  $E = 15$ .

Consider the energies  $E = 1, 5, 10, 15$ , and  $40$ . Try a few initial conditions for each energy. Determine visually whether the steady state behavior is regular or appears to be chaotic. Are there some values of  $E$  for which all the trajectories appear regular? Are there values of  $E$  for which all trajectories appear chaotic? Are there values of  $E$  for which both types of trajectories occur?

- Repeat your investigations of part (a), but plot the phase space diagrams  $p_1$  versus  $q_1$  and  $p_2$  versus  $q_2$ . Are these plots more useful for determining the nature of the trajectories than those drawn in part (a)?
- Draw the Poincaré plot with  $p_1$  plotted versus  $q_1$  only when  $q_2 = 0$  and  $p_2 > 0$ . Overlay trajectories from different initial conditions, but with the same total energy on the same plot. Duplicate the plot shown in Figure 6.13. Then produce Poincaré plots for the values of  $E$  given in part (a), with at least five different initial conditions per plot. Describe the different types of behavior.
- Is there a critical value of the total energy at which some chaotic trajectories first occur?
- Animate the double pendulum, showing the two masses moving back and forth. Describe how the motion of the pendulum is related to the behavior of the Poincaré plot.

Hamiltonian chaos has important applications in physical systems such as the solar system, the motion of the galaxies, and plasmas. It also has helped us understand the foundation for statistical mechanics. One of the most fascinating applications has been to quantum mechanics which has its roots in the Hamiltonian formulation of classical mechanics. A current area of interest is the quantum analogue of classical Hamiltonian chaos. The meaning of this analogue is not obvious because well-defined trajectories do not exist in quantum mechanics. Moreover, Schrödinger's equation is linear and can be shown to have only periodic and quasiperiodic solutions.

## 6.10 Perspective

As the many books and review articles on chaos can attest, it is impossible to discuss all aspects of chaos in a single chapter. We will revisit chaotic systems in Chapter 13 where we introduce the concept of fractals. We will find that one of the characteristics of chaotic dynamics is that the resulting attractors often have an intricate geometrical structure.

The most general ideas that we have discussed in this chapter are that *simple systems can exhibit complex behavior* and that chaotic systems exhibit *extreme sensitivity to initial conditions*. We also have learned that computers allow us to explore the behavior of dynamical systems and visualize the numerical output. However, the simulation of a system does not automatically lead to understanding. If you are interested in learning more about the phenomena of chaos and the associated theory, the suggested readings at the end of the chapter are a good place to start. We also invite you to explore chaotic phenomenon in more detail in the following projects.

## 6.11 Projects

The first several projects are on various aspects of the logistic map. These projects do not exhaust the possible investigations of the properties of the logistic map.

**Project 6.22.** A more accurate determination of  $\delta$  and  $\alpha$

We have seen that it is difficult to estimate  $\delta$  accurately by finding the sequence of values of  $b_k$  at which the trajectory bifurcates for the  $k$ th time. A better way to estimate  $\delta$  is to compute it from the sequence  $s_m$  of superstable trajectories of period  $2^{m-1}$ . We already have found that  $s_1 = 1/2$ ,  $s_2 \approx 0.80902$ , and  $s_3 \approx 0.87464$ . The parameters  $s_1, s_2, \dots$  can be computed directly from the equation

$$f^{(2^{m-1})}(x = \frac{1}{2}) = \frac{1}{2}. \quad (6.53)$$

For example,  $s_2$  satisfies the relation  $f^{(2)}(x = 1/2) = 1/2$ . This relation, together with the analytical form for  $f^{(2)}(x)$  given in (6.7), yields:

$$8r^2(1-r) - 1 = 0. \quad (6.54)$$

If we wish to solve (6.54) numerically for  $s_2$ , we need to be careful not to find the irrelevant solutions corresponding to a lower period. In this case we can factor out the solution  $r = 1/2$  and

| $m$ | period | $s_m$         |
|-----|--------|---------------|
| 1   | 1      | 0.500 000 000 |
| 2   | 2      | 0.809 016 994 |
| 3   | 4      | 0.874 640 425 |
| 4   | 8      | 0.888 660 970 |
| 5   | 16     | 0.891 666 899 |
| 6   | 32     | 0.892 310 883 |
| 7   | 64     | 0.892 448 823 |
| 8   | 128    | 0.892 478 091 |

Table 6.2: Values of the control parameter  $s_m$  for the superstable trajectories of period  $2^{m-1}$ . Nine decimal places are shown.

solve the resultant quadratic equation analytically to find  $s_2 = (1 + \sqrt{5})/4$ . Clearly  $r = s_1 = 1/2$  solves (6.54) with period 1, because from (6.53),  $f(1)(x = \frac{1}{2}) = 4r\frac{1}{2}(1 - \frac{1}{2}) = r = \frac{1}{2}$  only for  $r = 1/2$ .

1. It is straightforward to adapt the bisection method discussed in Section 6.6. Adapt the class `RecursiveFixedPointApp` to find the numerical solutions of (6.53). Good starting values for the left-most and right-most values of  $r$  are easy to obtain. The left-most value is  $r = r_\infty \approx 0.8925$ . If we already know the sequence  $s_1, s_2, \dots, s_m$ , then we can estimate  $\delta$  by

$$\delta_m = \frac{s_{m-1} - s_{m-2}}{s_m - s_{m-1}}. \quad (6.55)$$

We use this estimate for  $\delta_m$  to estimate the right-most value of  $r$ :

$$r_{\text{right}}^{(m+1)} = \frac{s_m - s_{m-1}}{\delta_m}. \quad (6.56)$$

We choose the desired precision to be  $10^{-16}$ . A summary of our results is given in Table 6.2. Verify these results and estimate  $\delta$ .

2. Use your values of  $s_m$  to obtain a better estimate of  $\alpha$ .

### Project 6.23. From chaos to order

The bifurcation diagram of the logistic map (see Figure 6.2) has many interesting features that we have not explored. For example, you might have noticed that there are several smooth dark bands in the chaotic region for  $r > r_\infty$ . Use `BifurcateApp` to generate the bifurcation diagram for  $r_\infty \leq r \leq 1$ . Note that the points are not uniformly distributed in each vertical line. For example, if we start at  $r = 1.0$  and decrease  $r$ , there is a band that narrows and eventually splits into two parts at  $r \approx 0.9196$ . If you look closely, you will see that the band splits into four parts at  $r \approx 0.899$ . In fact, if you look closely, you will see many more bands. What type of change occurs near the splitting (merging) of these bands)? Use `IterateMap` to look at the time series of  $x_{n+1} = f(x_n)$  for  $r = 0.9175$ . You will notice that although the trajectory looks random, it

oscillates back and forth between two bands. This behavior can be seen more clearly if you look at the time series of  $x_{n+1} = f^{(2)}(x_n)$ . A detailed discussion of the splitting of the bands can be found in Peitgen et al.

**Project 6.24.** Calculation of the Lyapunov spectrum

In Section 6.5 we discussed the calculation of the Lyapunov exponent for the logistic map. If a dynamical system has a multidimensional phase space, for example, the Hénon map and the Lorenz model, there is a set of Lyapunov exponents, called the Lyapunov spectrum, that characterize the divergence of the trajectory. As an example, consider a set of initial conditions that forms a filled sphere in phase space for the (three-dimensional) Lorenz model. If we iterate the Lorenz equations, then the set of phase space points will deform into another shape. If the system has a fixed point, this shape contracts to a single point. If the system is chaotic, then, typically, the sphere will diverge in one direction, but become smaller in the other two directions. In this case we can define three Lyapunov exponents to measure the deformation in three mutually perpendicular directions. These three directions generally will not correspond to the axes of the original variables. Instead, we must use a Gram-Schmidt orthogonalization procedure.

The algorithm for finding the Lyapunov spectrum is as follows:

1. Linearize the dynamical equations. If  $\mathbf{r}$  is the  $f$ -component vector containing the dynamical variables, then define  $\Delta\mathbf{r}$  as the linearized difference vector. For example, the linearized Lorenz equations are

$$\frac{d\Delta x}{dt} = -\sigma\Delta x + \sigma\Delta y \quad (6.57a)$$

$$\frac{d\Delta y}{dt} = -x\Delta z - z\Delta x + r\Delta x - \Delta y \quad (6.57b)$$

$$\frac{d\Delta z}{dt} = x\Delta y + y\Delta x - b\Delta z. \quad (6.57c)$$

2. Define  $f$  orthonormal initial values for  $\Delta\mathbf{r}$ . For example,  $\Delta\mathbf{r}_1(0) = (1, 0, 0)$ ,  $\Delta\mathbf{r}_2(0) = (0, 1, 0)$ , and  $\Delta\mathbf{r}_3(0) = (0, 0, 1)$ . Because these vectors appear in a linearized equation, they do not have to be small in magnitude.
3. Iterate the original and linearized equations of motion. One iteration yields a new vector from the original equation of motion and  $f$  new vectors  $\Delta\mathbf{r}_\alpha$  from the linearized equations.
4. Find the orthonormal vectors  $\Delta\mathbf{r}'_\alpha$  from the  $\Delta\mathbf{r}_\alpha$  using the Gram-Schmidt procedure. That is,

$$\Delta\mathbf{r}'_1 = \frac{\Delta\mathbf{r}_1}{|\Delta\mathbf{r}_1|} \quad (6.58a)$$

$$\Delta\mathbf{r}'_2 = \frac{\Delta\mathbf{r}_2 - (\Delta\mathbf{r}'_1 \cdot \Delta\mathbf{r}_2)\Delta\mathbf{r}'_1}{|\Delta\mathbf{r}_2 - (\Delta\mathbf{r}'_1 \cdot \Delta\mathbf{r}_2)\Delta\mathbf{r}'_1|} \quad (6.58b)$$

$$\Delta\mathbf{r}'_3 = \frac{\Delta\mathbf{r}_3 - (\Delta\mathbf{r}'_1 \cdot \Delta\mathbf{r}_3)\Delta\mathbf{r}'_1 - (\Delta\mathbf{r}'_2 \cdot \Delta\mathbf{r}_3)\Delta\mathbf{r}'_2}{|\Delta\mathbf{r}_3 - (\Delta\mathbf{r}'_1 \cdot \Delta\mathbf{r}_3)\Delta\mathbf{r}'_1 - (\Delta\mathbf{r}'_2 \cdot \Delta\mathbf{r}_3)\Delta\mathbf{r}'_2|}. \quad (6.58c)$$

It is straightforward to generalize the method to higher dimensional models.

5. Set the  $\Delta \mathbf{r}_\alpha(t)$  equal to the orthonormal vectors  $\Delta \mathbf{r}'_\alpha(t)$ .
6. Accumulate the running sum,  $S_\alpha$  as  $S_\alpha \rightarrow S_\alpha + \log |\Delta \mathbf{r}_\alpha(t)|$ .
7. Repeat steps (3)–(6) and periodically output the estimate of the Lyapunov exponents  $\lambda_\alpha = (1/n)S_\alpha$ , where  $n$  is the number of iterations.

To obtain estimates for the Lyapunov spectrum that represent the steady state attractor, only include data after the transient behavior has died out.

- a. Compute the Lyapunov spectrum for the Lorenz model for  $\sigma = 16$ ,  $b = 4$ , and  $r = 45.92$ . Try other values of the parameters and compare your results.
- b. Linearize the equations for the Hénon map and find the Lyapunov spectrum for  $a = 1.4$  and  $b = 0.3$  in (6.32).

**Project 6.25.** A spinning magnet

Consider a compass needle that is free to rotate in a periodically reversing magnetic field which is perpendicular to the axis of the needle. The equation of motion of the needle is given by

$$\frac{d^2 \phi}{dt^2} = -\frac{\mu}{I} B_0 \cos \omega t \sin \phi, \quad (6.59)$$

where  $\phi$  is the angle of the needle with respect to a fixed axis along the field,  $\mu$  is the magnetic moment of the needle,  $I$  its moment of inertia, and  $B_0$  and  $\omega$  are the amplitude and the angular frequency of the magnetic field, respectively. Choose an appropriate numerical method for solving (6.59), and plot the Poincaré map at time  $t = 2\pi n/\omega$ . Verify that if the parameter  $\lambda = \sqrt{2B_0\mu/I/\omega^2} > 1$ , then the motion of the needle exhibits chaotic motion. Briggs (see references) discusses how to construct the corresponding laboratory system and other nonlinear physical systems.

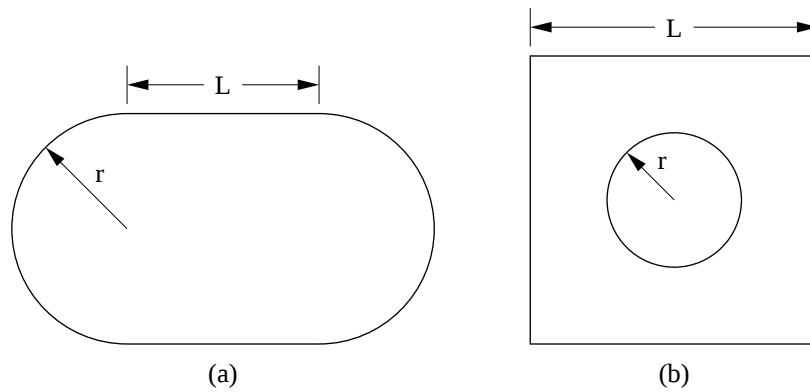


Figure 6.14: (a) Geometry of the stadium billiard model. (b) Geometry of the Sinai billiard model.

**Project 6.26.** Billiard models

Consider a two-dimensional planar geometry in which a particle moves with constant velocity along straight line orbits until it elastically reflects off the boundary. This straight line motion occurs in various “billiard” systems. A simple example of such a system is a particle moving with fixed speed within a circle. For this geometry the angle between the particle’s momentum and the tangent to the boundary at a reflection is the same for all points.

Suppose that we divide the circle into two equal parts and connect them by straight lines of length  $L$  as shown in Figure 6.14a. This geometry is called a stadium billiard. How does the motion of a particle in the stadium compare to the motion in the circle? In both cases we can find the trajectory of the particle by geometrical considerations. The stadium billiard model and a similar geometry known as the Sinai billiard model (see Figure 6.14b) have been used as model systems for exploring the foundations of statistical mechanics. There also is much interest in relating the behavior of a classical particle in various billiard models to the solution of Schrödinger’s equation for the same geometries.

a. Write a program to simulate the stadium billiard model. Use the radius  $r$  of the semicircles as the unit of length. The algorithm for determining the path of the particle is as follows:

1. Begin with an initial position  $(x_0, y_0)$  and momentum  $(p_{x0}, p_{y0})$  of the particle such that  $|\mathbf{p}_0| = 1$ .
2. Determine which of the four sides the particle will hit. The possibilities are the top and bottom line segments and the right and left semicircles.
3. Determine the next position of the particle from the intersection of the straight line defined by the current position and momentum, and the equation for the segment where the next reflection occurs.
4. Determine the new momentum,  $(p'_x, p'_y)$ , of the particle after reflection such that the angle of incidence equals the angle of reflection. For reflection off the line segments we have  $(p'_x, p'_y) = (p_x, -p_y)$ . For reflection off a circle we have

$$p'_x = [y^2 - (x - x_c)^2]p_x - 2(x - x_c)yp_y \quad (6.60a)$$

$$p'_y = -2(x - x_c)yp_x + [(x - x_c)^2 - y^2]p_y, \quad (6.60b)$$

where  $(x_c, 0)$  is the center of the circle. (Note that the momentum  $p_x$  rather than  $p'_x$  is on the right-hand side of (6.60b). Also remember that all lengths are scaled by the radius of the circle.)

5. Repeat steps (2)–(4).
- b. Determine if the particle dynamics is chaotic by estimating the largest Lyapunov exponent. One way to do so is to start two particles with almost identical positions and/or momenta (varying by say  $10^{-5}$ ). Compute the difference  $\Delta s$  of the two phase space trajectories as a function of the number of reflections  $n$ , where  $\Delta s$  is defined by

$$\Delta s = \sqrt{|\mathbf{r}_1 - \mathbf{r}_2|^2 + |\mathbf{p}_1 - \mathbf{p}_2|^2}. \quad (6.61)$$

Do the calculation for  $L = 1$ . The Lyapunov exponent can be found from a semilog plot of  $\Delta s$  versus  $n$ . Why does the exponential growth in  $\Delta s$  stop for sufficiently large  $n$ ? Repeat your calculation for different initial conditions and average your values of  $\Delta s$  before plotting. Repeat the calculation for  $L = 0.1, 0.5$ , and  $2.0$  and determine if your results depend on  $L$ .

- c. Another test for the existence of chaos is the reversibility of the motion. Reverse the momentum after the particle has made  $n$  reflections, and let the drawing color equal the background color so that the path can be erased. What limitation does roundoff error place on your results? Repeat this simulation for  $L = 1$  and  $L = 0$ .
- d. Place a small hole of diameter  $d$  in one of the circular sections of the stadium so that the particle can escape. Choose  $L = 1$  and set  $d = 0.02$ . Give the particle a random position and momentum, and record the time when the particle escapes through the hole. Repeat for at least  $10^4$  particles and compute the fraction of particles  $S(n)$  remaining after a given number of reflections  $n$ . The function  $S(n)$  will decay with  $n$ . Determine the functional dependence of  $S$  on  $n$ , and calculate the characteristic decay time if  $S(n)$  decays exponentially. Repeat for  $L = 0.1, 0.5$ , and  $2.0$ . Is the decay time a function of  $L$ ? Does  $S(n)$  decay exponentially for the circular billiard model ( $L = 0$ ) (see Bauer and Bertsch)?
- e. Choose an arbitrary initial position for the particle in a stadium with  $L = 1$ , and a small hole as in part (d). Choose at least 5000 values of the initial value  $p_{x0}$  uniformly distributed between 0 and 1. Choose  $p_{y0}$  so that  $|\mathbf{p}| = 1$ . Plot the escape time versus  $p_{x0}$ , and describe the visual pattern of the trajectories. Then choose 5000 values of  $p_{x0}$  in a smaller interval centered about the value of  $p_{x0}$  for which the escape time was greatest. Plot these values of the escape time versus  $p_{x0}$ . Do you see any evidence of self-similarity?
- f. Repeat steps (a)–(e) for the Sinai billiard geometry.

**Project 6.27.** Mode locking and the circle map

The driven, damped pendulum can be approximated by a one-dimensional difference equation for a range of amplitudes and frequencies of the driving force. This difference equation is known as the *circle map* and is given by

$$\theta_{n+1} = \left( \theta_n + \Omega - \frac{K}{2\pi} \sin 2\pi\theta_n \right). \quad (\text{modulo } 1) \quad (6.62)$$

The variable  $\theta$  represents an angle, and  $\Omega$  represents a frequency ratio, the ratio of the natural frequency of the pendulum to the frequency of the periodic driving force. The parameter  $K$  is a measure of the strength of the nonlinear coupling of the pendulum to the external force. An important quantity is the winding number which is defined as

$$W = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=0}^{m-1} \Delta\theta_n, \quad (6.63)$$

where  $\Delta\theta_n = \Omega - (K/2\pi) \sin 2\pi\theta_n$ .

- a. Consider the linear case,  $K = 0$ . Choose  $\Omega = 0.4$  and  $\theta_0 = 0.2$  and determine  $W$ . Verify that if  $\Omega$  is a ratio of two integers, then  $W = \Omega$  and the trajectory is periodic. What is the value

of  $W$  if  $\Omega = \sqrt{2}/2$ , an irrational number? Verify that  $W = \Omega$  and that the trajectory comes arbitrarily close to any particular value of  $\theta$ . Does  $\theta_n$  ever return exactly to its initial value? This type of behavior of the trajectory is termed *quasiperiodic*.

- b. For  $K > 0$ , we will find that  $W \neq \Omega$  and “locks” into rational frequency ratios for a range of values of  $K$  and  $\Omega$ . This type of behavior is called *mode locking*. For  $K < 1$ , the trajectory is either periodic or quasiperiodic. Estimate the value of  $W$  for  $K = 1/2$  and values of  $\Omega$  in the range  $0 < \Omega \leq 1$ . The widths in  $\Omega$  of the various mode-locked regions where  $W$  is fixed increase with  $K$ . Consider other values of  $K$ , and draw a diagram in the  $K$ - $\Omega$  plane ( $0 \leq K, \Omega \leq 1$ ) so that those areas corresponding to frequency locking are shaded. These shaded regions are called *Arnold tongues*.
- c. For  $K = 1$ , all trajectories are frequency-locked periodic trajectories. Fix  $K$  at  $K = 1$  and determine the dependence of  $W$  on  $\Omega$ . The plot of the  $W$  versus  $\Omega$  for  $K = 1$  is called the *Devil's staircase*.

**Project 6.28.** Chaotic scattering

In Chapter 5 we discussed the classical scattering of particles off a fixed target, and found that the differential cross section for a variety of interactions is a smoothly varying function of the scattering angle. That is, a small change in the impact parameter  $b$  leads to a small change in the scattering angle  $\theta$ . There is much current interest in cases where a small change in  $b$  leads to large changes in  $\theta$ . Such a phenomenon is called *chaotic scattering*, because of the sensitivity to initial conditions that is characteristic of chaos. The study of chaotic scattering is relevant to the design of electronic nanostructures, because many experimental structures exhibit this type of scattering.

A typical scattering model consists of a target composed of a group of fixed hard disks and a scatterer consisting of a point particle. We trace the path of the scatterer as it bounces off the disks, and measure  $\theta$  and the time of flight as a function of the impact parameter  $b$ . If a particle bounces inside the target region before leaving, the time of flight can be very long. There are even some trajectories for which the particle never leaves the target region.

Because it is difficult to monitor a trajectory that bounces back and forth between the hard disks, we instead consider a two-dimensional map that contains the key features of chaotic scattering (see Yalcinkaya and Lai for further discussion). The map is given by

$$x_{n+1} = a \left[ x_n - \frac{1}{4}(x_n + y_n)^2 \right], \quad (6.64a)$$

and

$$y_{n+1} = \frac{1}{a} \left[ y_n + \frac{1}{4}(x_n + y_n)^2 \right], \quad (6.64b)$$

where  $a$  is a parameter. The target region is centered at the origin. In an actual scattering experiment, the relation between  $(x_{n+1}, y_{n+1})$  and  $(x_n, y_n)$  would be much more complicated, but the map (6.64) captures most of the important features of realistic chaotic scattering experiments. The iteration number  $n$  is analogous to the number of collisions of the scattered particle off the disks. When  $x_n$  or  $y_n$  is significantly different from zero, the scatterer has left the target region.

- a. Write a program to iterate the map (6.64). Let  $a = 8.0$  and  $y_0 = -0.3$ . Choose  $10^4$  initial values of  $x_0$  uniformly distributed on the interval  $0 < x_0 < 0.1$ . Determine the time  $T(x_0)$ ,

the number of iterations for which  $x_n \leq -5.0$ . After this time,  $x_n$  rapidly moves to  $-\infty$ . Plot  $T(x_0)$  versus  $x_0$ . Then choose  $10^4$  initial values in a smaller interval centered about a value of  $x_0$  for which  $T(x_0) > 7$ . Plot these values of  $T(x_0)$  versus  $x_0$ . Do you see any evidence of self-similarity?

- b. A trajectory is said to be *uncertain* if a small change  $\epsilon$  in  $x_0$  leads to a change in  $T(x_0)$ . We expect that the number of uncertain trajectories,  $N$ , will depend on a power of  $\epsilon$ , that is,  $N \sim \epsilon^\alpha$ . Determine  $N(\epsilon)$  for  $\epsilon = 10^{-p}$  with  $p = 2$  to  $7$  using the values of  $x_0$  in part (a). Then determine the uncertainty dimension  $1 - \alpha$  from a log-log plot of  $N$  versus  $\epsilon$ . Repeat these measurements for other values of  $a$ . Does  $\alpha$  depend on  $a$ ?
- c. Choose  $4 \times 10^4$  initial conditions in the same interval as in part (a) and determine the number of trajectories,  $S(n)$ , that have not yet reached  $x_n = -5$  as a function of the number of iterations  $n$ . Plot  $\ln S(n)$  versus  $n$  and determine if the decay is exponential. It is possible to obtain algebraic decay for values of  $a$  less than approximately  $6.5$ .
- d. Let  $a = 4.1$  and choose 100 initial conditions uniformly distributed in the region  $1.0 < x_0 < 1.05$  and  $0.60 < y_0 < 0.65$ . Are there any trajectories that are periodic and hence have infinite escape times? Due to the accumulation of roundoff error, it is possible to find only finite, but very long escape times. These periodic trajectories form closed curves, and the regions enclosed by them are called KAM surfaces.

### Project 6.29. Chemical reactions

In Project 4.17 we discussed how chemical oscillations can occur when the reactants are continuously replenished. In this project we introduce a set of chemical reactions that exhibits the period doubling route to chaos. Consider the following reactions (see Peng et al.):



Each of the above reactions has an associated rate constant. The time dependence of the concentrations of  $A$ ,  $B$ , and  $C$  is given by:

$$\frac{dA}{dt} = k_1 P + k_2 PC - k_3 A - k_4 AB^2 \quad (6.66a)$$

$$\frac{dB}{dt} = k_3 A + k_4 AB^2 - k_5 B \quad (6.66b)$$

$$\frac{dC}{dt} = k_4 B - k_5 C. \quad (6.66c)$$

We assume that  $P$  is held constant by replenishment from an external source. We also assume the chemicals are well mixed so that there is no spatial dependence. In Section 7.8 we discuss the

effects of spatial inhomogeneities due to molecular diffusion. Equations (6.65) can be written in a dimensionless form as

$$\frac{dX}{d\tau} = c_1 + c_2 Z - X - XY^2 \quad (6.67a)$$

$$c_3 \frac{dY}{d\tau} = X + XY^2 - Y \quad (6.67b)$$

$$c_4 \frac{dZ}{d\tau} = Y - Z, \quad (6.67c)$$

where the  $c_i$  are constants,  $\tau = k_3 t$ , and  $X$ ,  $Y$ , and  $Z$  are proportional to  $A$ ,  $B$ , and  $C$ , respectively.

- Write a program to solve the coupled differential equations in (6.67). We suggest using a fourth-order Runge-Kutta algorithm with an adaptive step size. Plot  $\ln Y$  versus the time  $\tau$ .
- Set  $c_1 = 10$ ,  $c_3 = 0.005$ , and  $c_4 = 0.02$ . The constant  $c_2$  is the control parameter. Consider  $c_2 = 0.10$  to  $0.16$  in steps of  $0.005$ . What is the period of  $\ln Y$  for each value of  $c_2$ ?
- Determine the values of  $c_2$  at which the period doublings occur for as many period doublings as you can determine. Compute the constant  $\delta$  (see (6.9)) and compare its value to the value of  $\delta$  for the logistic map.
- Make a bifurcation diagram by taking the values of  $\ln Y$  from the Poincaré plot at  $X = Z$  and plotting them versus the control parameter  $c_2$ . Do you see a sequence of period doublings?
- Use three-dimensional graphics to plot the trajectory of (6.67) with  $\ln X$ ,  $\ln Y$ , and  $\ln Z$  as the three axes.

Describe the attractors for some of the cases considered in part (a). chaos!period doubling!chemical reactions—)

## Appendix 6A: Stability of the Fixed Points of the Logistic Map

In the following, we derive analytical expressions for the fixed points of the logistic map. The fixed-point condition is given by

$$x^* = f(x^*). \quad (6.68)$$

From (6.5) this condition yields the two fixed points

$$x^* = 0 \quad \text{and} \quad x^* = 1 - \frac{1}{4r}. \quad (6.69)$$

Because  $x$  is restricted to be positive, the only fixed point for  $r < 1/4$  is  $x = 0$ . To determine the stability of  $x^*$ , we let

$$x_n = x^* + \epsilon_n, \quad (6.70a)$$

and

$$x_{n+1} = x^* + \epsilon_{n+1}. \quad (6.70b)$$

Because  $|\epsilon_n| \ll 1$ , we have

$$\begin{aligned} x_{n+1} &= f(x^* + \epsilon_n) \approx f(x^*) + \epsilon_n f'(x^*) \\ &= x^* + \epsilon_n f'(x^*). \end{aligned} \quad (6.71)$$

If we compare (6.70b) and (6.71), we obtain

$$\epsilon_{n+1}/\epsilon_n = f'(x^*). \quad (6.72)$$

If  $|f'(x^*)| > 1$ , the trajectory will diverge from  $x^*$  because  $|\epsilon_{n+1}| > |\epsilon_n|$ . The opposite is true for  $|f'(x^*)| < 1$ . Hence, the local stability criteria for a fixed point  $x^*$  are

1.  $|f'(x^*)| < 1$ ,  $x^*$  is stable;
2.  $|f'(x^*)| = 1$ ,  $x^*$  is marginally stable;
3.  $|f'(x^*)| > 1$ ,  $x^*$  is unstable.

If  $x^*$  is marginally stable, the second derivative  $f''(x)$  must be considered, and the trajectory approaches  $x^*$  with deviations from  $x^*$  inversely proportional to the square root of the number of iterations.

For the logistic map the derivatives at the fixed points are respectively

$$f'(x = 0) = \left. \frac{d}{dx} [4rx(1-x)] \right|_{x=0} = 4r, \quad (6.73)$$

and

$$f'(x = x^*) = \left. \frac{d}{dx} [4rx(1-x)] \right|_{x=1-1/4r} = 2 - 4r. \quad (6.74)$$

It is straightforward to use (6.73) and (6.74) to find the range of  $r$  for which  $x^* = 0$  and  $x^* = 1 - 1/4r$  are stable.

If a trajectory has period two, then  $f^{(2)}(x) = f(f(x))$  has two fixed points. If you are interested, you can solve for these fixed points analytically. As we found in Problem 6.2, these two fixed points become unstable at the same value of  $r$ . We can derive this property of the fixed points using the chain rule of differentiation:

$$\left. \frac{d}{dx} f^{(2)}(x) \right|_{x=x_0} = \left. \frac{d}{dx} f(f(x)) \right|_{x=x_0} = f'(f(x_0))f'(x) \Big|_{x=x_0}. \quad (6.75)$$

If we substitute  $x_1 = f(x_0)$ , we can write

$$\left. \frac{d}{dx} f(f(x)) \right|_{x=x_0} = f'(x_1)f'(x_0). \quad (6.76)$$

In the same way, we can show that

$$\left. \frac{d}{dx} f^{(2)}(x) \right|_{x=x_1} = f'(x_0)f'(x_1). \quad (6.77)$$

We see that if  $x_0$  becomes unstable, then  $|f^{(2)'}(x_0)| > 1$  as does  $|f^{(2)'}(x_1)|$ . Hence,  $x_1$  also is unstable at the same value of  $r$ , and we conclude that both fixed points of  $f^{(2)}(x)$  bifurcate at the same value of  $r$ , leading to an trajectory of period 4.

From (6.74) we see that  $f'(x = x^*) = 0$  when  $r = 1/2$  and  $x^* = 1/2$ . Such a fixed point is said to be *superstable*, because as we found in Problem 6.4, convergence to the fixed point is relatively rapid. In general, superstable trajectories occur whenever one of the fixed points is at  $x^* = 1/2$ .

## Appendix 6B: Finding the Roots of a Function

The roots of a function  $f(x)$  are the values of the variable  $x$  for which the function  $f(x)$  is zero. Even an apparently simple equation such as

$$f(x) = \tan x - x - c = 0. \quad (6.78)$$

where  $c$  is a constant cannot be solved analytically for  $x$ .

Whatever the function and whatever the approach to root finding, the first step should be to learn as much as possible about the function. We will plot the functions to help us guess the approximate locations of the roots.

Newton's (or the Newton-Raphson) method is based on replacing the function by the first two terms of the Taylor expansion of  $f(x)$  about the root  $x$ . If our initial guess for the root is  $x_0$ , we can write  $f(x) \approx f(x_0) + (x - x_0)f'(x_0)$ . If we set  $f(x)$  equal to zero and solve for  $x$ , we find  $x = x_0 - f(x_0)/f'(x_0)$ . If we have made a good choice for  $x_0$ , the resultant value of  $x$  should be closer than  $x_0$  to the root. The general procedure is to calculate successive approximations as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (6.79)$$

If this series converges, it converges very quickly. However, if the initial guess is poor or if the function has closely spaced multiple roots, the series may not converge. The successive iterations of Newton's method is an example of a map. Newton's method also works with complex functions as we will see in the following problem.

### Problem 6.30. Cube roots

Consider the function  $f(z) = z^3 - 1$ , where  $z = x + iy$ , and  $f'(z) = z^2$ . Map the range of convergence of (6.79) in the region  $[-2 < x < 2, -2 < y < 2]$  in the complex plane. Color the starting  $z$  value red, green, or blue depending on the root to which the initial guess converges. If the trajectory does not converge, color the starting point black. For more insight add a mouse handler to your program so that if you click on your plot, the sequence of iterations starting from the point where you clicked will be shown.

The following problem discusses a situation that typically arises in courses on quantum mechanics.

**Problem 6.31.** Energy levels in a finite square well

The quantum mechanical energy levels in the one-dimensional finite square well can be found by solving the relation:

$$\epsilon \tan \epsilon = \sqrt{\rho^2 - \epsilon^2}, \quad (6.80)$$

where  $\epsilon = \sqrt{mEa^2/2\hbar}$  and  $\rho = \sqrt{mV_0a^2/2\hbar}$  are defined in terms of the particle mass  $m$ , the particle energy  $E$ , the width of the well  $a$ , and the depth of the well  $V_0$ . The function  $\epsilon \tan \epsilon$  has zeros at  $\epsilon = (0, \pi, 2\pi, \dots)$  and asymptotes at  $\epsilon = (0, \pi/2, 3\pi/2, 5\pi/2, \dots)$ . The function  $\sqrt{\rho^2 - \epsilon^2}$  is a quarter circle of radius  $\rho$ . Write a program to plot these two functions with  $\rho = 3$  and then use Newton's method to determine the roots of (6.80). Find the value of  $\rho$  and thus  $V_0$ , such that below this value there is only one energy level and above this value there is more than one. At what value of  $\rho$  do three energy levels first appear?

In Section 6.6 we introduced the bisection root finding algorithm. This algorithm is implemented in the `Root` class in the numerics package. It can be used with any function.

Listing 6.6: The `bisection` method is defined in the `Root` class in the numerics package.

```
public static double bisection(final Function f, double x1, double x2, final double tolerance) {
    int count = 0;
    int maxCount = (int) (Math.log(Math.abs(x2 - x1)/tolerance)/Math.log(2));
    maxCount = Math.max(MAX_ITERATIONS, maxCount) + 2;
    double y1 = f.evaluate(x1), y2 = f.evaluate(x2);
    if (y1 * y2 > 0) { // y1 and y2 must have opposite sign
        return Double.NaN; // interval does not contain a root
    }
    while (count < maxCount) {
        double x = (x1 + x2) / 2;
        double y = f.evaluate(x);
        if (Math.abs(y) < tolerance) return x;
        if (y * y1 > 0) { // replace the end-point that has the same sign
            x1 = x;
            y1 = y;
        }
        else {
            x2 = x;
            y2 = y;
        }
        count++;
    }
    return Double.NaN; // did not converge in max iterations
}
```

There are many root-finding techniques but all can be made to fail with appropriately chosen functions. The bisection algorithm is guaranteed to converge if you can find an interval where the function changes sign. However, it is slow. Newton's algorithm is very fast, but may not converge. We design an algorithm in the following problem that combines these two approaches.

**Problem 6.32.** Finding roots

Modify Newton's algorithm to keep track of the interval between the most negative and the most positive values of  $x$  while iterating. If the solution jumps outside this interval, interrupt Newton's method and use the bisection algorithm for one iteration. Test the root at the end of the iterative process to check that the algorithm actually found a root. Test your algorithm on the function in (6.78).

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