

Chapter 18

Seeing in Special and General Relativity

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We compute how objects appear at relativistic speeds and in the vicinity of a large spherically symmetric mass.

18.1 Special Relativity

How do objects appear at relativistic speeds? The Lorentz-Fitzgerald length contraction in the direction of motion is not the only effect. A single observer forms an image of an object by collecting light from the entire object. When the single observer sees the object, she does not see its current position nor its true shape, but sees each part of the object where it was when the light was emitted. Because of the finite speed of light, we need to calculate when and where along the object's trajectory each light ray originated to determine the image seen by a single observer.

The relative velocity of an object with respect to a single observer defines a direction, which we choose to be the direction of the x -axis in the observer's frame of reference, S . Let S' be the rest system of the object and v be the velocity of S with respect to S' ; the origins coincide at $t = t' = 0$. The Lorentz transformation connecting S and S' is

$$x' = \gamma(x - vt) \tag{18.1a}$$

$$y' = y \tag{18.1b}$$

$$z' = z \tag{18.1c}$$

$$t' = \gamma(t - vx/c), \tag{18.1d}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. In the rest frame of the object, the separation between two points on the object is

$$d' = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}. \tag{18.2}$$

In the rest frame of the observer, the separation is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \quad (18.3)$$

where

$$x_2 - x_1 = \gamma(x'_2 - x'_1) \quad (18.4a)$$

$$y_2 - y_1 = y'_2 - y'_1 \quad (18.4b)$$

$$z_2 - z_1 = z'_2 - z'_1. \quad (18.4c)$$

The change in the x separation is known as the Lorentz-Fitzgerald contraction. If we know the shape of the object in the rest frame, we can compute the shape in the observer's frame by applying an affine transformation (see Chapter 17) that rescales the object's x dimension by γ . Listing 18.1 shows how this transformation is done using a two-dimensional wire frame model of a ring. The `ContractedRing` class defines a ring of unit radius in the object's rest frame using an array of `Point2D` objects. These points are transformed into the observer's frame and the transformed shape is drawn by connecting the points using line segments.

Listing 18.1: The `ContractedRing` class implements the Lorentz-Fitzgerald contraction of a ring moving in the x direction.

```
package org.opensourcephysics.sip.ch18;
import java.awt.*;
import java.awt.geom.*;
import org.opensourcephysics.display.*;

public class ContractedRing implements Drawable {
    double vx = 0, time = 0;
    Point2D[] labPts, pixPts;
    public ContractedRing(double x0, double y0, double vx, int numPts) {
        labPts = new Point2D[numPts];
        pixPts = new Point2D[numPts];
        double theta = 0, dtheta = 2*Math.PI/(numPts-1);
        // unit radius circle
        for(int i = 0; i < numPts; i++) {
            double x = Math.cos(theta); // x coordinate
            double y = Math.sin(theta); // y coordinate
            labPts[i] = new Point2D.Double(x, y);
            theta += dtheta;
        }
        this.vx = vx;
        // Lorentz-Fitzgerald contraction
        AffineTransform at = AffineTransform.getScaleInstance(Math.sqrt(1-vx*vx), 1);
        at.transform(labPts, 0, labPts, 0, labPts.length);
        // translate to initial position
        at = AffineTransform.getTranslateInstance(x0, y0);
        at.transform(labPts, 0, labPts, 0, labPts.length);
    }
}
```

```

public void setTime(double t) {
    double dt = t-time;
    // convert position to position at new time
    AffineTransform at = AffineTransform.getTranslateInstance(vx*dt, 0);
    at.transform(labPts, 0, labPts, 0, labPts.length);
    time = t;
}

void drawShape(DrawingPanel panel, Graphics2D g2) {
    // convert from lab coordinates to pixels
    AffineTransform at = panel.getPixelTransform();
    at.transform(labPts, 0, pixPts, 0, labPts.length);
    g2.setColor(Color.RED);
    for(int i = 1, n = labPts.length; i<n; i++) {
        g2.draw(new Line2D.Double(pixPts[i-1], pixPts[i]));
    }
}

public void draw(DrawingPanel panel, Graphics g) {
    Graphics2D g2 = (Graphics2D) g;
    drawShape(panel, g2);
}
}

```

Exercise 18.1. Lorentz-Fitzgerald contraction

Write a small test program that instantiates and displays a `ContractedRing` object. Measure the dimensions of the on-screen object to verify that the Lorentz-Fitzgerald contraction is computed correctly.

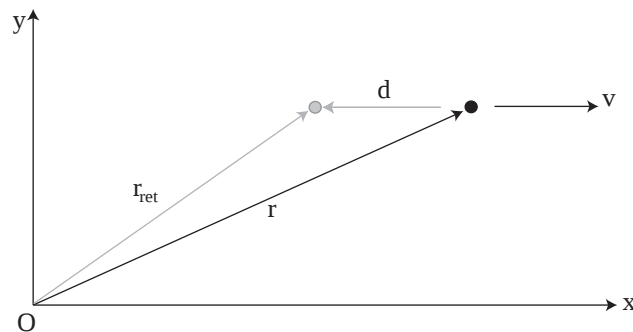


Figure 18.1: The geometry used to derive the retarded position and time for an arbitrary point moving with constant velocity in the x direction.

We now introduce retardation effects. Let $\mathbf{r} = (x, y)$ be the current location of an arbitrary point on the object as shown in Figure 18.1. Because of the finite speed of light, the observer sees

this point not at its current location, but at a previous location

$$\mathbf{r}_{\text{ret}} = (x - \delta, y). \quad (18.5)$$

The x coordinate is *retarded* by $\delta = v\tau$, where τ is the travel time from $\mathbf{r}_{\text{ret}}$ to the observer. The distance from the retarded position to the observer is

$$r_{\text{ret}} = \sqrt{(x - \delta)^2 + y^2}. \quad (18.6)$$

We use the speed of light to convert distance to light travel time $r_{\text{ret}} = c\tau$, substitute for δ , and obtain

$$c\tau = \sqrt{(x - v\tau)^2 + y^2}. \quad (18.7)$$

If we square both sides and apply the quadratic formula, we find

$$\tau = \frac{-x\beta + \sqrt{x^2\beta^2 + (1 - \beta^2)(x^2 + y^2)}}{(1 - \beta^2)}, \quad (18.8)$$

where $\beta = v/c$; we chose the positive square root to make the travel time τ positive.

We can incorporate (18.8) into our program to obtain the image seen by a stationary observer. We subclass `ContractedRing` and add methods to calculate and draw the retarded positions of the points on the ring. Retarded points are stored in an array and the retarded positions are calculated by solving (18.8). The complete class is shown in Listing 18.2.

Listing 18.2: The `ObservedRing` class models the appearance of a ring traveling in the x direction at relativistic speeds.

```
package org.opensourcephysics.sip.ch18;
import java.awt.*;
import java.awt.geom.*;
import org.opensourcephysics.display.*;

public class ObservedRing extends ContractedRing {
    Point2D[] retardPts;
    public ObservedRing(double x0, double y0, double vx, int numPts) {
        super(x0, y0, vx, numPts);
        retardPts = new Point2D[numPts];
        for(int i = 0; i < numPts; i++) {
            retardPts[i] = new Point2D.Double();
        }
    }

    void setRetardedPts() {
        double gammaSquared = (1 - vx*vx);
        for(int i = 0, n = labPts.length; i < n; i++) {
            double x = labPts[i].getX();
            double y = labPts[i].getY();
            double tau = (-vx*x + Math.sqrt(x*x*vx*vx + gammaSquared*(x*x + y*y)))/gammaSquared;
            retardPts[i].setLocation(x - vx*tau, y);
        }
    }
}
```

```

    }
}

void drawObservedShape(DrawingPanel panel, Graphics2D g2) {
    setRetardedPts();
    // convert from view to pixel coordinates
    AffineTransform at = panel.getPixelTransform();
    at.transform(retardPts, 0, pixPts, 0, retardPts.length);
    g2.setColor(Color.BLACK);
    for(int i = 1, n = retardPts.length; i < n; i++) {
        g2.draw(new Line2D.Double(pixPts[i-1], pixPts[i]));
    }
}

public void draw(DrawingPanel panel, Graphics g) {
    Graphics2D g2 = (Graphics2D) g;
    drawShape(panel, g2);
    drawObservedShape(panel, g2);
}
}

```

Exercise 18.2. Relativistic ring

Write a test program that instantiates and displays the apparent shape of a rapidly moving ring. Explain the sharp convex point when the front edge of the ring touches the observer. Does the ring ever appear to be concave? Why?

Exercise 18.3. Relativistic ruler

Modify the `SRSShape` class to display a long narrow rectangle. How easy is it for an observer to “see” the Lorentz-Fitzgerald contraction if this ruler is moving along the x axis? Use the coordinate display to measure the apparent length of the ruler at various positions.

Exercise 18.4. Relativistic square

Write a small test program that instantiates and displays the apparent shape of a square moving at $y = b$ past an observer. Does the square remain a square? Demonstrate that the observer can see the square’s hidden side. This effect is known as *Terrell rotation*.

If the center of a moving ring travels along the x axis, then the shape seen in the simulation also represents the view of a three-dimensional sphere approaching an observer head-on. This special case was treated analytically by Suffern, but most other two- and three-dimensional objects cannot be treated analytically and are best visualized using the help of a computer. A complete visualization is quite complicated because other effects, such as the Doppler effect, aberration, and angular changes in the intensity distribution of the emitted light, also need to be taken into account to produce an accurate representation of a rapidly moving object.

18.2 General Relativity

The idea that space can be curved was first tested by Gauss who measured the interior angles of a large triangle by placing lanterns on three mountaintops. Although Gauss obtained the Euclidian result of 180° , a later experiment by Eddington using starlight showed that the sum of the interior angles is not 180° . It is an experimental fact that the universe is indeed non-Euclidian.

The Eddington experiment was remarkable in that it confirmed Einstein's general theory of relativity and showed that space and time are not separable absolute properties. We cannot measure space, only distances between events in space using rulers, light beams, and clocks. Furthermore, the separation between events is not the same for different observers unless they incorporate both spatial and temporal separations into their definition of distance. In the absence of gravitational fields, observers moving at constant relative velocity can reconcile (18.2) and (18.3) and obtain the same "distance" only if they agree that distance between events, $\Delta\sigma$, includes time and is measured as

$$(\Delta\sigma)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2, \quad (18.9)$$

where c is the speed of light, Δt is the temporal separation, and Δx , Δy , and Δz are the coordinate separations. Equation (18.9) is based on Einstein's special theory of relativity and is known as the Minkowski metric. It follows from Einstein's assumption that Maxwell's equations must be the same for all observers in uniform relative motion and leads naturally to the equivalence of mass and energy embodied in the famous equation $E = mc^2$.

Einstein's second great insight enabled him to incorporate gravity into the spacetime fabric by generalizing (18.9). He did so by assuming that acceleration and gravitation are indistinguishable. Imagine an elevator compartment resting on the surface of Earth in which the occupants perform experiments, such as dropping objects or swinging pendula, that reveal the presence of Earth's gravity. Then the occupants are placed in a compartment far from any gravitational object. The compartment is accelerating at 9.8 m/s^2 . According to Einstein, the two experimental results must be identical. Furthermore, if the near-Earth elevator cable is cut to produce a freely falling reference frame, then the occupants will be unable to detect the gravitational field. The implication is that we can do away with gravity and regard it as a consequence of an accelerated reference frame in four-dimensional spacetime. It took Einstein ten years to incorporate this equivalence of gravitational forces and accelerated motion into the special theory of relativity to produce the general theory.

Einstein's general theory of relativity produces ten independent non-linear differential equations. Calculations using this theory are truly daunting and require sophisticated mathematical techniques such as tensor analysis and Riemannian geometry. All forms of energy gravitate and nonlinearities arise because a body's gravitational field is itself a form of energy and therefore gravitates. Few analytical results are known. Two of the most important are the dynamical equations for particles and light in the vicinity of a spherically symmetric mass. Except for very special cases or very weak fields, these dynamical equations must be solved numerically to predict how the particles move and how they appear when seen by an observer.

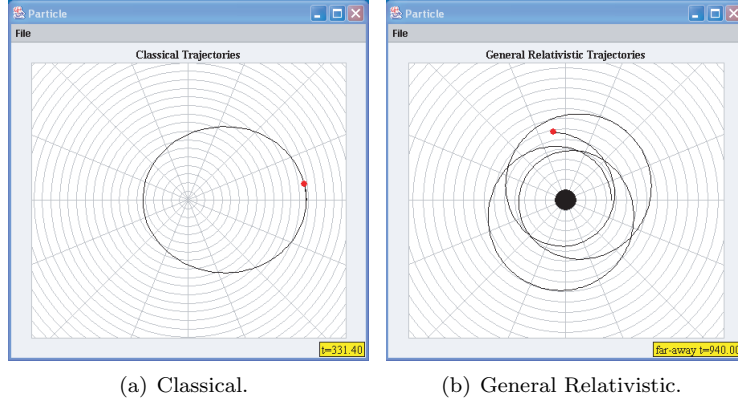


Figure 18.2: Comparison of classical and general relativistic particle trajectories in the vicinity of a spherically symmetric mass.

18.3 Polar Coordinates

General relativistic trajectories of particles and light in the vicinity of spherically symmetric gravitational fields are conveniently described using polar coordinates. We therefore start by reformulating the classical two-body problem using polar coordinates. If the motion is confined to a plane, rectangular coordinates, (x, y) , and polar coordinates, (r, ϕ) , are related by

$$x = r \cos \phi, \quad y = r \sin \phi \quad (18.10)$$

and

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan \frac{y}{x}. \quad (18.11)$$

The radial velocity is given by

$$\dot{r} = \frac{dr}{dt} = \frac{\vec{r} \cdot \vec{v}}{r}, \quad (18.12)$$

and the angular velocity is given by

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{mr^2}, \quad (18.13)$$

where L is the magnitude of the angular momentum, $L = |\vec{r} \times \vec{p}|$.

To construct the appropriate differential equations, the radial and angular accelerations can be obtained by differentiating (18.12) and (18.13) with respect to time. Another approach is to use the Lagrangian

$$\mathcal{L} = \frac{1}{2} [\dot{r}^2 + r^2 \dot{\phi}^2] + \frac{GM}{r} \quad (18.14)$$

and apply Lagrange's equations of motion.

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (18.15)$$

If we do the differentiation, we obtain the following rate for the polar state vector $(r, \dot{r}, \phi, \dot{\phi}, t)$.

$$\frac{dr}{dt} = \dot{r} \quad (18.16a)$$

$$\frac{d\dot{r}}{dt} = r \left(\frac{d\phi}{dt} \right)^2 - \frac{GM}{r^2} \quad (18.16b)$$

$$\frac{d\phi}{dt} = \dot{\phi} \quad (18.16c)$$

$$\frac{d\dot{\phi}}{dt} = -\frac{2}{r} \frac{d\phi}{dt} \frac{dr}{dt} \quad (18.16d)$$

$$\frac{dt}{dt} = 1. \quad (18.16e)$$

Exercise 18.5. Classical trajectories

Modify the `PlanetApp` program discussed in Chapter 5 so that the classical trajectory of a particle is calculated using the polar rate (18.16). Use (18.10)–(18.13) to determine the initial state and compare your results with those of the `PlanetApp` program.

The default Open Source Physics plotting panel contains an axis object that displays a cartesian coordinate grid. This axes can be replaced with a polar grid; the method `createPolarType2` does such a replacement (see Figure 18.2):

```
PlottingPanel plottingPanel = PlottingPanel.createPolarType2("Trajectory",1.0);
```

The first parameter is the plot's title and the second parameter is the radial grid separation. The new plotting panel also displays polar coordinates in the bottom left when the mouse is clicked or dragged.

Exercise 18.6. Polar Coordinates

Modify Exercise 18.5 so that polar coordinates are displayed.

18.4 Black Holes and Schwarzschild Coordinates

The classic text, *Exploring Black Holes* by Edwin Taylor and John Wheeler discusses in detail the physics near objects that have undergone gravitational collapse. Such an object is known as a *black hole* because not even light can escape from its vicinity. We will calculate the general relativistic trajectories of particles and light near a spherically symmetric gravitational mass. Because physical space is non-Euclidian, a two-dimensional plot of these trajectories will be distorted. Unlike the simulations of classical orbits, the orbits drawn appear quite different when seen by a viewer standing in the real world. We must, in fact, calculate the trajectories of multiple light rays in order to construct the view as seen by a single observer.

Because time is incorporated as a fourth dimension and because space is curved, a general relativistic coordinate system centered on a spherically symmetric mass is more complicated than a Euclidean three-dimensional coordinate system. The azimuthal angle, ϕ , can still be defined as the ratio of the arc length to the circumference on an imaginary circle because the spherically symmetric gravitational mass, M , is located at the origin. The radial coordinate, however, is not defined as the physical distance from the center. Rather, it is calculated using a path that circumnavigates the central mass.

$$r = \text{circumference}/(2\pi) \quad (18.17)$$

The time coordinate is defined using a wristwatch located far from the center of attraction. Note the *non-local* character of these spacetime coordinates. The wristwatch worn by the surveyor circumnavigating the mass in order to measure r is not the time used to record events at that value of r . This spacetime coordinate system is referred to as *Schwarzschild coordinates* in honor of the physicist Karl Schwarzschild and is merely a universal bookkeeping device that enables us to translate observations from one reference frame into another reference frame. Schwarzschild coordinates give rise to a metric, known as the Schwarzschild metric, that enables us to calculate a four-dimensional distance between adjacent spacetime events. This metric is

$$d\sigma^2 = -d\tau^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\phi^2, \quad (18.18)$$

where t , r , and ϕ refer to the faraway time, the radial coordinate, and the azimuthal coordinate, respectively. Because we associate distance with a positive number, it is common to use $d\sigma^2$ when the right-hand side of (18.18) is positive and to use $d\tau^2$ when the right-hand side of (18.18) is negative. As in special relativity, these two forms are referred to as the space-like form and the time-like form of the metric, respectively. Note that both time and distance have units of length in (18.18). The speed of light, c , is the conversion factor,

$$t_{\text{meters}} = ct_{\text{seconds}}. \quad (18.19)$$

Note also that mass has units of meters. The conversion factor is given in terms of the speed of light and Newton's gravitational constant, G .

$$M_{\text{meters}} = \frac{G}{c^2} M_{\text{kg}}. \quad (18.20)$$

If we freeze time, so that $dt = 0$, then the Schwarzschild metric predicts that two simultaneous events far from the central mass are separated by the Euclidian result expressed in polar coordinates

$$d\sigma^2 = dr^2 + r^2d\phi^2. \quad (18.21)$$

Strange things happen if two events are close to the gravitational mass. The separation (known as the proper distance) between two adjacent events becomes

$$d\sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\phi^2. \quad (18.22)$$

The proper distance $d\sigma$ is the distance measured by a surveyor placing meter sticks in space between two locations. This distance is clearly greater than the result predicted by (18.21) when

events occur at different r -coordinates. In fact, the rate of change of proper length with respect to the r -coordinate becomes infinite as we approach what is known as the event horizon, $r = 2M$. Distance around a gravitational mass has no such singularity, which is why this distance is used to define the r -coordinate. (The singularity at the event horizon is an artifact of the Schwarzschild coordinate system. An object falling into a black hole passes through the event horizon without incident and is only crushed at $r = 0$.)

Exercise 18.7. Distance

Although the rate of change of distance with respect to r becomes infinite, the distance from a point outside the horizon to the horizon is finite. Write a short program that integrates $d\sigma$ from a point $r = a$ outside the horizon to a point arbitrarily close to the horizon. What is the distance from $r = 4$ to $r = 2$ when $M = 1$?

Exercise 18.8. Mass

- a. Although general relativity correctly predicts the shape of orbits around any spherically symmetric gravitational object, not all objects have an event horizon. (Black holes do.) The event horizon assumes that the mass of the entire object is within the horizon and this assumption clearly implies very high mass densities. Calculate the r coordinate of the event horizon for an object having the mass of the Earth and compare it to the radius of the Earth. Repeat the calculation for the Sun.
- b. The event horizon for the black hole believed to exist at the center of our galaxy has an event horizon of $r = 7.6 \times 10^9$ m. What is its mass in units of a solar mass?

The variable t in the Schwarzschild metric is time as measured by a faraway observer. Time as measured by a local observer is known as the proper time, τ . Observers experience time as measured by their wristwatches and (18.18) shows that wristwatch time interval $d\tau$ depends on location. A faraway observer measuring the time between two light flashes records a value of dt while an observer standing next to these flashes measures a time interval $d\tau$ given by

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2. \quad (18.23)$$

Proper time intervals near a gravitational mass are clearly smaller than faraway time intervals. This result gives rise to the gravitational red shift when applied to light.

Exercise 18.9. Time

Estimate the difference due to gravitational effects between local time and faraway time during one hour for an observer standing on Earth. Does special relativity play a role in an actual measurement?

18.5 Particle and Light Trajectories

The physics describing the trajectory of a particle in the vicinity of a gravitational mass can be formulated using the principle of extremal aging (see Hanc). This principle states that a particle

takes a path through spacetime such that the time recorded by a wristwatch attached to the particle τ is a minimum. Because Lagrangian dynamics is based on the principle that the integral of the Lagrangian over time (called the action) also is an extremum, we construct a Lagrangian using the Schwarzschild metric.

$$\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}) = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 \right]^{1/2}. \quad (18.24)$$

This Lagrangian must satisfy (18.15). We take the required derivatives and simplify terms and obtain the following system of first-order differential equations:

$$\frac{dr}{dt} = \dot{r} \quad (18.25a)$$

$$\frac{d\dot{r}}{dt} = \frac{4M^3 - 4M^2r - 4M^2r^3\dot{\phi}^2 + 4Mr^4\dot{\phi}^2 - r^5\dot{\phi}^2 + r^2(M - 3M\dot{r}^2)}{(2M - r)r^3} \quad (18.25b)$$

$$\frac{d\phi}{dt} = \dot{\phi} \quad (18.25c)$$

$$\frac{d\dot{\phi}}{dt} = \frac{2(-3M + r)\dot{r}\dot{\phi}}{(2M - r)r} \quad (18.25d)$$

$$\frac{dt}{dt} = 1. \quad (18.25e)$$

Note that the independent variable in 18.25 is faraway time. The metric provides an additional differential equation if we wish to track the particle's proper time, τ .

$$\frac{d\tau}{dt} = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 \right]^{1/2}. \quad (18.26)$$

Exercise 18.10. General relativistic trajectories

- Write a program that plots the general relativistic trajectory of a particle using Schwarzschild coordinates. Verify that you obtain circular orbits for $v = \sqrt{M/r}$ for $r \geq 6M$.
- Show that there are no stable circular orbits for $r < 6M$.
- Add the differential equation for proper time. What is the proper time for one complete orbit at $r = 6$? This interval is the orbital period as measured by an observer traveling with the particle. Compare this wristwatch orbital period to the faraway orbital period and to the time interval predicted by (18.23). Explain any discrepancies in your numerical values.
- Perturb the circular orbit at $r = 9$ by giving the particle an initial tangential velocity of $v = 0.345c$. At what rate does the perihelion of the orbit advance?

The equations for light can be obtained by adding a constraint to (18.24) using a Lagrange multiplier. This constraint is the condition that the proper length along a light path be zero.

$$0 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\phi^2, \quad (18.27)$$

If we add the Lagrange multiplier, do the differentiation, and simplify terms, we obtain a rate equation that can be solved using standard numerical techniques. (The use of a computer algebra program would be helpful.)

$$\frac{dr}{dt} = \dot{r} \quad (18.28a)$$

$$\frac{d\dot{r}}{dt} = \frac{-4M^2 + 2Mr + (r - 5M)r^3\dot{\phi}^2}{r^3} \quad (18.28b)$$

$$\frac{d\phi}{dt} = \dot{\phi} \quad (18.28c)$$

$$\frac{d\dot{\phi}}{dt} = \frac{2(-3M + r)\dot{r}\dot{\phi}}{(2M - r)r} \quad (18.28d)$$

$$\frac{dt}{dt} = 1. \quad (18.28e)$$

Exercise 18.11. Light trajectories

- Write a program that plots the general relativistic trajectory of light using Schwarzschild coordinates. Demonstrate the deflection of star light passing near a gravitational mass by plotting the trajectory of a light ray.
- Verify that light orbits a black hole at $r = 3$ and $M = 1$.
- Show that a gravitational mass can act as a lens by plotting the trajectory of two light rays that leave a point source at different angles but later cross. The two light rays should pass on opposite sides of the mass. Do the two light beams always arrive at the crossing point at the same time?

18.6 Seeing

Because of the non-linearity of the Schwarzschild metric, simulation plays an essential role. A calculation of a view of the stars in the vicinity of a black hole, for example, would require the solution of the light-ray trajectory for angles within the eye's field of view.

Angles drawn on a Schwarzschild map are not the same as angles seen by an observer because distances on the map are distorted. A stationary observer at a constant r -value is known as a *shell observer* because he is standing on a stationary shell at fixed (r, ϕ) coordinates. Launch angles measured by such a shell observer can easily be converted to angles on the Schwarzschild map by taking into account the contraction by $\sqrt{1 - 2M/r}$ in the radial direction.

$$\tan \theta_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} \tan \theta_{\text{schw}}. \quad (18.29)$$

Use this transformation in Exercise 18.12 and in the *Seeing Orbits* project.

Exercise 18.12. Knife edge trajectory

Many important properties of light rays can be expressed in terms of an impact parameter, b , defined as

$$b = r \left(1 - \frac{2M}{r}\right)^{-1/2} \sin \theta_{\text{shell}}. \quad (18.30)$$

For example, light that is launched with $b = \sqrt{27}M$ enters an unstable orbit that teeters between an escape to infinity and a plunge into the black hole. Demonstrate this effect including sensitivity to initial conditions and numerical roundoff errors. What will a shell observer see if he looks into space at an angle that has this impact parameter?

Problem 18.13. Seeing

Imagine a grid of light beacons located far away from a black hole in the $\phi = \pi$ direction. A shell observer at $\phi = 0$ with an arbitrary value of r attempts to view the grid by looking toward the black hole. What will he see? One way to answer this question is to assume a reasonable field of view (try 180°) for the eye and calculate light rays leaving the eye at equal angular intervals. Compute the light paths and tabulate where the ray crosses the beacon grid as a function of angle. Because it is unlikely that the light rays will intersect a beacon location, use interpolation to determine the angles at which beacons appear. Plot these locations to show the observer's view.

18.7 General Relativistic Dynamics

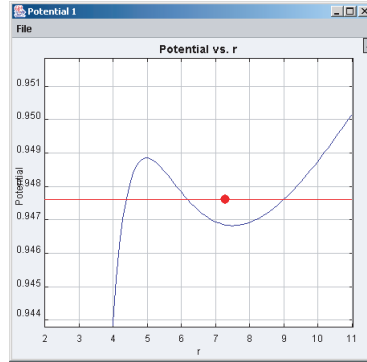


Figure 18.3: The effective potential, $V(r)$, of a particle in the vicinity of a black hole.

In general relativity, the angular momentum, L , of a particle of mass m is

$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau}, \quad (18.31)$$

and the energy, E , is

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}. \quad (18.32)$$

We solve (18.31) for $d\phi$ and (18.32) for dt and substitute the result into the time-like form of the metric and obtain a relation for $dr/d\tau$:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{L}{mr}\right)^2\right]. \quad (18.33)$$

In analogy with the classical effective potential function for a particle moving in a gravitational field, we use (18.33) to define a *relativistic effective potential* (see Figure 18.3).

$$\left(\frac{V(r)}{m}\right)^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{L}{mr}\right)^2\right]. \quad (18.34)$$

Exercise 18.14. Energy and angular momentum

Show that energy and angular momentum are conserved for the orbits you observed in Exercise 18.10.

Exercise 18.15. Effective potential

Add a plot of the effective potential, $V(r)$, to Exercise 18.10. Add a horizontal line showing the energy per unit mass and place a red marker on this line showing the particle's position. Describe the effective potential and the motion of the marker when the orbit is circular, when the orbit precesses, and when the orbit plunges toward the event horizon.

18.8 The Kerr Metric

Because almost all astronomical objects rotate, most black holes likely have angular momentum. The metric for a spinning black hole was derived by Kerr in 1964. For simplicity, we show the metric for particle motion in the equatorial plane. Note that this metric contains a new angular momentum parameter, a .

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}dtd\phi - \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1}dr^2 - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right)r^2d\phi^2. \quad (18.35)$$

Because there are two values at which the coefficient of dr^2 increases without limit, $r_h = M \pm \sqrt{M^2 - a^2}$, there are two horizons. Furthermore, we see that the largest real value of a consistent with real values of r_h is $a = M$. This maximum value of a limits the angular momentum of a black hole. Because we are interested in maximizing the effect of rotation, we simplify (18.35) by letting the angular momentum parameter take on its maximum value. The metric for this extreme Kerr black hole is

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}dtd\phi - \left(1 - \frac{M}{r}\right)^{-2}dr^2 - R^2d\phi^2, \quad (18.36)$$

where

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r}. \quad (18.37)$$

We recast this metric as a Lagrangian and follow a derivation by Hanc and Tuleja, and obtain the rate:

$$\frac{dr}{dt} = \dot{r} \quad (18.38a)$$

$$\begin{aligned} \frac{d\dot{r}}{dt} = & -\frac{(M-r)^2(M-2M^2\dot{\phi}+M^3\dot{\phi}^2-r^3\dot{\phi}^2)}{r^4} \\ & + \frac{2M^3-2M^4\dot{\phi}+3Mr^2-M^2r(1+6r\dot{\phi})}{r^2(M-r)^2}\dot{r}^2 \end{aligned} \quad (18.38b)$$

$$\frac{d\phi}{dt} = \dot{\phi} \quad (18.38c)$$

$$\frac{d\dot{\phi}}{dt} = \frac{4M^3\dot{\phi}-2M^4\dot{\phi}^2+6Mr^2\dot{\phi}-2r^3\dot{\phi}-2M^2(1+3r^2\dot{\phi}^2)}{r^2(M-r)^2}\dot{r} \quad (18.38d)$$

$$\frac{dt}{dt} = 1. \quad (18.38e)$$

Exercise 18.16. Falling into a spinning black hole

Write a program that plots the general relativistic trajectory of a particle near an extreme black hole using (18.38).

- Follow the trajectory of a particle that starts from rest far from the center of the extreme black hole. Describe the trajectory.
- A particle is thrown with an angular momentum opposite to the hole's spin.

A space ship near a black hole must fire its rockets tangentially as well as radially to keep from being swept along in the direction of a spinning black hole. This effect, known as *frame dragging*, occurs near any spinning gravitational object including the Earth. The effect becomes dramatic as the ship approaches the horizon for the extreme black hole, $r_h = M$. (The horizon is where the metric coefficient of dr^2 becomes infinite.) Note that the coefficient of the dt^2 term goes to zero at $r = r_s = 2M$. This value is called the static limit. The space between the static limit and the horizon is dragged along in the direction of rotation of the black hole so that an observer cannot remain at a fixed angle no matter how powerful her rockets.

18.9 Projects

Project 18.17. Three-dimensional rapidly moving objects

Extend the analysis in Section 18.1 to three-dimensional objects and model the appearance of 3D objects as seen by a single observer at the origin using the transformation and rendering techniques described in Chapter 17. Does a sphere appear to be a sphere even when it passes by an observer? Does a cube appear to be a cube?

Project 18.18. Light Links

Imagine two stationary observers near a black hole wishing to establish a communication link using a laser beam. In what direction should the laser be pointed to establish such a link? Simulate this scenario using two draggable objects on a Schwarzschild map and draw the light ray representing the communication link. Use a root finding algorithm, such as the bisection method introduced in Chapter 6, to determine the proper launch angle. Calculate and display the proper distance along this light path and study how this distance changes as the light path grazes the event horizon.

Construct a light triangle connecting three observers. Display the sum of the interior angles as measured by the observers to simulate Gauss's mountain top experiment.

Project 18.19. Seeing Orbits

Viewing an orbit requires that we calculate the particle's trajectory and the trajectory of the light ray from the particle to the viewer. An added complication arises because the light reaching the view is retarded by the travel time. Write a program that shows an orbiting particle as seen by a stationary observer in the equatorial plane by keeping track of both particle and light-link parameters. Extend this program for an observer standing above a pole.

18.10 References

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