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Optimal Investment Horizons

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Abstract. In stochastic finance, one traditionally considers the return as a competitive measure of an asset, *i.e.*, the profit generated by that asset after some fixed time span Δt, say one week or one year. This measures how well (or how bad) the asset performs over that given period of time. It has been established that the distribution of returns exhibits "fat tails" indicating that large returns occur more frequently than what is expected from standard Gaussian stochastic processes [1,2,3]. Instead of estimating this "fat tail" distribution of returns, we propose here an alternative approach, which is outlined by addressing the following question: What is the smallest time interval needed for an asset to cross a fixed return level of say 10%? For a particular asset, we refer to this time as the *investment horizon* and the corresponding distribution as the *investment horizon distribution*. This latter distribution complements that of returns and provides new and possibly crucial information for portfolio design and risk-management, as well as for pricing of more exotic options. By considering historical financial data, exemplified by the Dow Jones Industrial Average, we obtain a novel set of probability distributions for the investment horizons which can be used to estimate the optimal investment horizon for a stock or a future contract.

PACS.

Financial data have been recorded for a long time as they represent an invaluable source of information for statistical investigations of financial markets. In the early days of stochastic finance it was argued that the distribution of returns (see definition below) of an asset should follow a normal (Gaussian) distribution [4,5]. However, by analysing large, and often high-frequency, financial data sets, it has been established that these distributions on short time scales — typically less then a month, or so — can posses so-called "fat-tails", *i.e.* distributions that show strong deviations from that of a Gaussian [1,2,3] with higher probabilities for large events. This is similar to the distributions found for turbulence in air and fluids which have led to comparisons between the statistics of financial markets and that of turbulent fluids [5,6,7,8]. In turbulence, one obtains stretched exponential distributions which find their analogy in finance when considering higher order correlations of the asset price [9,10].

In order to get a deeper understanding of the fluctuation of financial markets it is important to supplement this established information of fluctuations in the returns with alternative measures. In the present paper, we therefore ask the following "inverse" question: "What is the typical time span needed to generate a fluctuation or a movement (in the price) of a given size" [11,12,13,14,15,16]. Given a fixed log-return barrier, ρ, of a stock or an index as well as a fixed investment date, the corresponding time span is estimated for which the log-return of the stock or index *for the first time* reaches the level ρ. This can also be called the first passage time through the level (or barrier) [14, 15,16,17] ρ. As the investment date runs through the past (price) history of the stock or index, the accumulated values of the first passage times form the probability distribution function of the investment horizons for the smallest time period needed in the past to produce a log-return of at least magnitude ρ. The maximum of this distribution determines the most probable investment horizon which therefore is the optimal investment horizon for that given stock or index.

The first passage time is important from an economic point of view in several ways. Firstly, say an investor plans to sell or buy a certain asset. Then, of course, he or she is interested in doing the transaction at a point in time that will optimize the potential profit, *i.e.* to sell for the highest possible price, or, for a buyer, to buy for the lowest price. However, the problem is that one does not know when the price is optimal. Therefore, the best one can do, from a statistical point of view, is to make a transaction at a time that is probabilistically favorable. This optimal time, as we will see, is determined by the maximum of the first passage time distribution, *i.e. the most likely first passage time*. Secondly, for a holder of an European type option,

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either a call or a put of given strike price, the most likely first passage time will, in much the same way as presented above, define the optimal maturity of the option. Furthermore, for an American type call option the most likely first passage time of the underlying asset will be useful to know when to exercise the option. These same arguments apply even more to exotic options used in the financial industry [18]. Thirdly, the investment distribution for *negative* levels of returns, provides crucial information for the implementation of certain *stop-loss* strategies. Finally, but not least, the first passage distribution will by itself give invaluable, non-trivial information about the stochasticity of the underlying asset price. This point has recently been demonstrated explicitly in the (related) field of turbulence [11,12].

To illustrate our procedure we consider the Dow Jones Industrial Average (DJIA). We analyze the daily closure of the DJIA from its beginning on May 26, 1896 to June 5, 2001 (present). This leave us with 105 years of data, or almost 30 000 trading days. The data set to be analysed is depicted in Fig. 1. The log-return over a time interval Δt of an asset of price $S(t)$, at time t, is defined as

$$r_{\Delta t}(t) = s(t+\Delta t) - s(t), \tag{1}$$

where $s(t) = \ln S(t)$, *i.e.*, the log-return is just the log-price change of the asset. The investment horizon, $\tau_\rho(t)$, at time t, for a return level ρ is defined at the smallest time interval, Δt, that satisfies the relation $r_{\Delta t}(t) \geq \rho$, or in mathematical terms [14] $\tau_\rho(t) = \inf\{\Delta t > 0 \,|\, r_{\Delta t}(t) \geq \rho\}$. The investment horizon distribution, $p(\tau_\rho)$, is obtained as the histogram of investment horizons τ_ρ. Furthermore, we introduce the cumulative distribution for the horizon being larger then τ_ρ, *i.e.*

$$P(\tau_\rho) = \int_{\tau_\rho}^{\infty} p(t)\, dt. \tag{2}$$

It is well-known, and easy to see from Fig 1, that there is a substantial upward drift with time in the DJIA which is an indication of the overall growth of the world economy. In order to reduce this drift, we have wavelet filtered [19] the data on a scale larger then 1000 trading days, and we will therefore in this study limit ourselves to the behavior up-to 1000 trading days (about 4 years). This is achieved by first transforming the log-price $s(t) = \ln S(t)$ to the wavelet-domain, setting all wavelet coefficient corresponding to scales larger then 1000 trading days to zero, and finally transforming back to the time domain. This procedure, which results in the filtered log-price time series $\tilde{s}(t)$, will reduce the effect of drift as can be seen explicitly from Fig. 1. In the following analysis we will therefore use $s(t)$ to denote the *filtered* time series, and subsequently $S(t) = \exp(s(t))$ for the filtered price.

To present our analysis we show in Fig. 2 the investment horizon distribution $p(\tau_\rho)$ for a reasonably large value of the return level, $\rho = 0.05$ (*i.e.* 5%). This figure indicates that the obtained distribution has a very interesting and unexpected form. It exhibits a rather well defined and pronounced maximum, followed by an extended tail for very long time horizons indicating a non-zero and important probability of large passage times (note that the τ_ρ-axis is logarithmic). We expect that these long investment horizons reflect periods where the market is reasonably calm and quiet, or is going down for a long period of time before finally coming back up again. The short horizons on the other hand – those around the maximum – are observed in more volatile periods, which appears to be the most common scenario. Indeed the maximum obtained at $\tau_\rho = \tau_\rho^*$ is the most likely horizon, which we call the *optimal investment horizon*. This kind of distribution of the investment horizon statistics for economical markets has, to the best of our knowledge, not been published before.

To better understand the tail of this distribution, we consider a rather small level of return ρ. If this level is small enough, it is likely that the return will break through the level after the first day, while larger horizons will become more an more unlikely. However, the probability for a large horizon will not be zero; if, say, we consider a positive (but still small) level ρ, then a period of recession will result in a $\tau_\rho(t)$ that might be considerably larger then one day since it takes time to recover after a set back. For instance, after the 1927 stock market crash, it took more then two decades for the DJIA to regain the value it had just before the crash. In the limit $\rho \to 0^+$ the horizon distribution $p(\tau_0)$ is known in the literature as the *first return* probability distribution for the underlying stochastic process [14,15]. In Fig. 3 (lowest solid curve) the cumulative distribution, $P(\tau_0)$, for the DJIA data set is shown. It is observed that the tail of this distribution scales as a power law, $P(\tau_0) \sim \tau_0^{-\alpha_0}$, with $\alpha_0 \simeq 1/2$, over almost three orders of magnitudes in time. This value of the exponent can be understood as follows: If we assume, as is common in the financial literature [4,5], that the asset price, $S(t)$, can be well approximated by a geometrical Brownian motion, then trivially it follows that $s(t) = \ln S(t)$ will just be an ordinary Brownian motion. From the literature it is well-known that the first-return probability distribution of a fractional Brownian motion scales like [14,15] $p(\tau) \sim \tau^{H-2}$, where H is the Hurst exponent, and hence $P(\tau) \sim \tau^{H-1}$. Since the Hurst exponent of an ordinary Brownian motion is $H = 1/2$, the empirically observed scaling (see Fig. 3) $P(\tau_0) \sim \tau_0^{-1/2}$ is a consequence of the (at least close to) geometrical Brownian motion behavior of $S(t)$. This argument of an unbiased geometrical Brownian motion is also strengthen by observing that $P(\tau_0 = 1) \simeq 1/2$, meaning that the log-price change over one day raises half of the time and drops in the remaining half. It should noticed that in order to observe the power-law of exponent $\alpha_0 = 1/2$ for small levels, the filtered data have to be used. Using the raw data would, due to the presence of the drift, result in a slightly larger (smaller) exponent for a positive (negative) small level ρ.

Fig. 3 also shows the cumulative distributions $P(\tau_\rho)$ for different choices of the return level ρ, *i.e.*, $\rho = 0.01$, 0.02, 0.05, 0.10 and 0.20. From this figure it is seen that the tail exponent, α_ρ, is rather insensitive to the return level. In particular one finds that $\alpha_\rho \simeq 1/2$ over a broad range of values for ρ, a value that is consistent with the geometrical

Brownian motion hypothesis of the underlying asset price process (see Eq. (3) below). Moreover, it is observed that as the level ρ is increased from zero, the most likely horizon moves away form $\tau_0 = 1$ and toward larger values. In other words — there is an *optimal investment horizon*, τ^*_ρ, corresponding to, and depending on, a given level of return ρ. Furthermore, we have checked, and found, that there is an approximate symmetry under $\rho \to -\rho$ for the investment distribution as long as the filtered data are used. One therefore does not have to consider negative levels explicitly. On the other hand, if the analysis is based on the raw data there is a clear asymmetry.

The time needed for a general time series to reach a certain level is in the mathematical literature known as the *first passage problem.* For a Brownian motion this problem has been solved analytically [15,16]

$$p(t) = \frac{1}{\sqrt{\pi}} \frac{a}{t^{3/2}} \exp\left(-\frac{a^2}{t}\right). \qquad (3)$$

where a is proportional to the level ρ. For large times one recovers, from Eq. (3), the well known distribution of first return times to the origin $p(t) \sim t^{-3/2}$. In order to fit a functional form to the distribution in Fig. 2 we generalize this expression and suggest the following form

$$p(t) = \frac{\nu}{\Gamma\left(\frac{\alpha}{\nu}\right)} \frac{\beta^{2\alpha}}{(t+t_0)^{\alpha+1}} \exp\left\{-\left(\frac{\beta^2}{t+t_0}\right)^\nu\right\}, \qquad (4)$$

which reduces to Eq. (3) in the limit when $\alpha = 1/2$, $\beta = a$, $\nu = 1$, and $t_0 = 0$, since $\Gamma(1/2) = \sqrt{\pi}$. The form (4) seems to be a good approximation to the empirical investment horizon distributions. The shift t_0 is needed in order to fit the optimal horizon well, and its actual value may depend on possible short-scale drift. The full-drawn curve in Fig. 2 shows a (maximum likelihood) fit to the empirical data with the functional form Eq. (4), and the agreement is observed to be excellent (the fitted parameter values are given in the caption of Fig. 2).

As the optimal horizon provides an important information for an investor, a relevant question would now be: How does the optimal horizon, τ^*_ρ, depend on the return level ρ ? This dependence, as measured from the empirical horizon distribution, is shown in Fig. 4. Intuitively, it is clear that the optimal horizon will increase rather rapidly as the return is increased. However, we observe that this increase occurs in a systematic fashion

$$\tau^*_\rho \sim \rho^\gamma, \qquad (5)$$

with $\gamma \simeq 1.8$, see Fig. 4. For a Brownian motion, with a first passage distribution Eq. (3), it is straightforward to derive that the power law exponent should be $\gamma = 2$ for the *whole* range of ρ. Not surprisingly, we therefore find a deviation from standard theories for the variation of the optimal horizon with the return level. Furthermore, we systematically find $\nu > 1$ when $\rho \neq 0$, which is also inconsistent with the geometrical Brownian hypothesis.

In conclusion, we have obtained a new set of distributions of investment horizons for returns of a prescribed level. We have found that the (empirical) optimal investment horizon depends on the return level in a systematic fashion, which is not consistent with the the geometrical Brownian motion hypothesis typically assumed in theoretical finance [4,5]. The obtained distributions as well as the variation of the optimal horizon can be applied if one wants to estimate the most probable time period needed to stay in the market if an investor aims at a specific optimal return. Similar passage time distributions are found in turbulence of fluids (where they are called inverse structure functions). It indicates that these distribution functions of passage times could be a general and important feature of systems which exhibit extreme events like for example in finance, turbulence, earthquakes, and avalanches in granular media.

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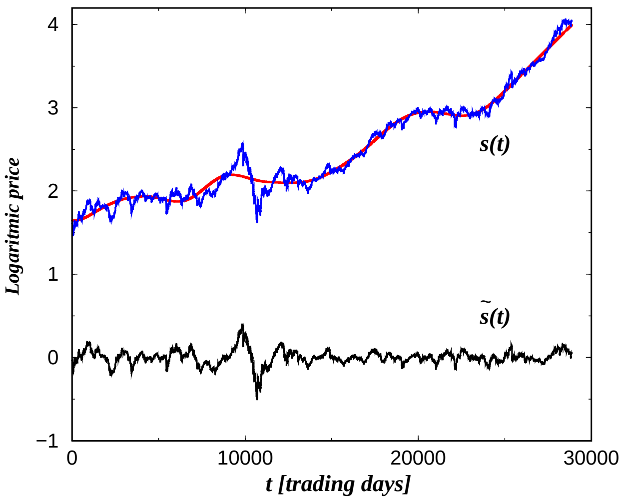

Fig. 1. The historic daily logarithmic closure prices of the Dow Jones Industrial Average (DJIA) over the period from May 26, 1896 to June 5, 2001. The upper curly curve is the raw logarithmic DJIA price $s(t) = \ln S(t)$, where $S(t)$ is the historic daily closure prices of the DJIA. The smooth curve represents the drift on a scale larger then 1000 trading days. This drift was obtained by a wavelet technique as described in the main text. The lower curly curve represents the wavelet filtered logarithmic DJIA data, $\tilde{s}(t)$. Those filtered data are just the fluctuation of $s(t)$ around the drift.

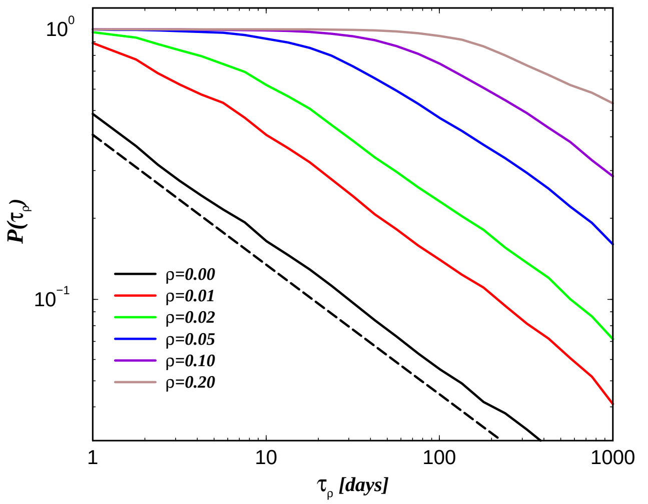

Fig. 3. The empirical cumulative probability distributions (solid lines), $P(\tau_\rho)$, vs. horizon τ_ρ for different levels $\rho = 0$, 0.01, 0.02, 0.05, 0.10, and 0.20 (from bottom to top), for the (wavelet filtered) Dow Jones Industrial Average. The dashed line corresponds to the geometrical Brownian motion assumption for the underlying price process, in which case one should have $P(\tau) \sim \tau^{-\alpha}$ with a tail index $\alpha = 1/2$. For the level $\rho = 0$ the geometrical Brownian motion assumption seems reasonable (*i.e* $\alpha_o \simeq 1/2$), while the tail index, α_ρ, shows only a weak (if any) dependence on the level ρ.

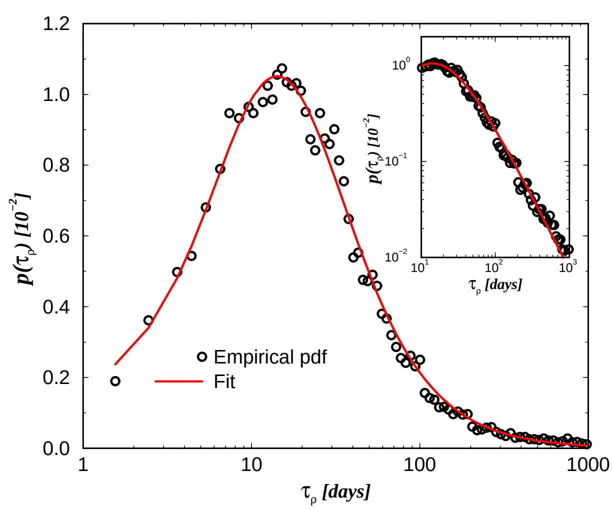

Fig. 2. The probability distribution function (pdf), $p(\tau_\rho)$, of the investment horizons (first passage times) measured in trading days, τ_ρ, at a level $\rho = 0.05$ (*i.e.* 5% return). The data used to produce this figure are the wavelet filtered logarithmic returns calculated from the historic daily closure prices (Fig. 1). The open circles represents the empirical pdf (at level ρ). We notice the pronounced maximum of this function at approximately $\tau_\rho^* = 15$ trading days (note the log-linear scale). This maximum represents the most probable time of producing a return of 5%. The solid line represents a maximum likelihood fit to the functional form (4) with parameters: $\alpha = 0.50$, $\beta = 4.5$ days$^{1/2}$, $\nu = 2.4$, and $t_0 = 11.2$ days. The inset is the same figure on log-log scale such that the power-law behavior of the tail is more easily observed.

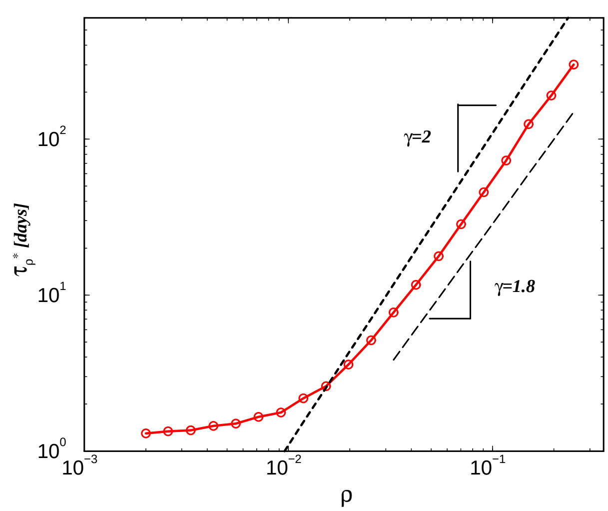

Fig. 4. The optimal investment horizon, τ_ρ^*, as a function of the log-return level ρ. The open circles represent the empirical results obtained from the (wavelet filtered) historical Dow Jones data. The dashed line, of slope $\gamma = 2$, corresponds to the geometrical Brownian motion hypothesis for the underlying asset price. One observes that for small levels this hypothesis fails dramatically. However, for levels of the order of a few percents or larger, the geometrical Brownian motion assumption becomes more realistic, *but* also in this region a discrepancy is observed. Empirically one finds a scaling $\tau_\rho^* \sim \rho^\gamma$ with $\gamma \simeq 1.8$ (long dashed line).