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Synchronization of multi-mode pulse coupled stochastic oscillators[♣]

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It is known that an ensemble of two-mode pulse-emitting stochastic oscillators globally coupled through their collective output will synchronize for some controllable parameter values. These oscillators proved to be appropriate for modeling the dynamics of rhythmic applause or several other synchronization phenomena in biological systems. Within this work we generalize this two-mode oscillator model to the case of several modes and investigate the effect on their synchronization. Computer simulation proves that synchronization appears for several modes as well and the system undergoes a phase transition-like phenomena while changing a controllable parameter. However, if the number of modes is increased synchronization will be present for smaller and smaller parameter range.

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1. Introduction

Synchronization phenomena is observed in many physical, social, biological and chemical systems, and it has attracted the interest of scientists for centuries [1]. It is an interesting example of self organization. It is also important to mention that in many biological and sociological systems synchronization is not the primary aim of the individuals, and synchronization appears as a co-product of some optimization phenomenon. Some well-known examples in such sense are: (a) the synchronized clapping of the spectators after a good performance [2], [3]; (b) synchronization of fireflies and cricket chirps [4]; (c) synchronization of neurons fire [5]; (d) synchronization of pacemaker cells in the heart [6]; (e) or even synchronization of the menstrual cycles of women living together [7]. In the present work we will study a special type of synchronization which is induced by a simple collective optimization phenomenon in multi-mode pulse-coupled oscillators.

2. Classical models for synchronization

There are two main category of models widely used for describing synchronization of non-identical oscillators. One is for phase-coupled oscillators (Kuramoto-type models), and the other one is for pulse-coupled oscillators (integrate and fire-type models). For a more detailed review of these models we suggest references [8], [9], and [10]. Here only a very brief description will be given.

The Kuramoto model [8] is a mathematical model for the collective coupling of a large set of non-identical oscillators. It is motivated by the behaviour of many chemical and biological oscillators, and it has found widespread applications. In the most popular version of the Kuramoto model each of the N oscillators is considered to have its own intrinsic natural frequency ω_i, and it is coupled equally to all other oscillators. Surprisingly, this fully nonlinear model can be solved exactly for some special coupling in the infinite-N limit. This is done by a clever transformation and the application of self-consistency arguments. The main result of this model is that for each ensemble of oscillators there is a critical coupling level, above which partial synchronization of the system will appear. This critical coupling is linearly proportional with the dispersion of the oscillators natural frequencies [8].

The integrate-and-fire oscillators is possibly the simplest model of neuronal behavior [9,10]. A variable associated with the membrane voltage of a neuron is allowed to increase from zero until a threshold is reached. Once the threshold is attained, the oscillator is said to "fire". This variable is instantaneously reset to zero and the process repeats. Once fired, the neuron must recover before firing again. The slow collection and quick release of voltage is called integrate and fire behavior. Although it is a gross approximation of neural activity, the integrate-and-fire model has been of heuristic value to neurobiology. The Fitzhugh-Nagumo equations were derived simultaneously by Fitzhugh and Nagumo for such oscillators [11]. These equations provide a fairly detailed picture of the action potential of a neuron and can also lead to synchronization phenomena.

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In order to describe the dynamics of rhythmic applause and synchronization in several other biological systems Nikitin et. al studied pulse-coupled stochastic oscillator systems with two different oscillation modes [12]. Depending on the global output level, the cycle of these oscillators can be performed in two ways: $A \to B_I \to C \to A$ or $A \to B_{II} \to C \to A$, respectively. The periods corresponding to these two modes T_I and T_{II} are given as $T_I = \tau_A + \tau_{B_I} + \tau_C$ and $T_{II} = \tau_A + \tau_{B_{II}} + \tau_C$, where τ_A, τ_B, and τ_C are time intervals spent in states *A*, *B*, and *C*, respectively.

The stochastic part of the dynamics is *state A*, and τ_A is a stochastic variable with probability density:

$$P(\tau_A) = \frac{1}{\tau^*} e^{-\frac{\tau_A}{\tau^*}} \quad (1)$$

($\tau^* = \langle \tau_A \rangle$). State *A* should be imagined and modeled with an escape dynamics of a stochastic field-driven particle from a potential valley of depth *U*. If the stochastic force field is totally uncorrelated with $\langle \xi \rangle = 0$ and $\langle \xi(t)\xi(t+\tau) \rangle = D\delta(\tau)$ we get a the distribution of escape times given in (1) with $\tau^* \propto e^{\frac{U}{D}}$. In analogy with the well-known Fitzhugh-Nagumo system, state *A* corresponds to a stochastic reaction time of the neuron fire. This causes all the experimentally observed fluctuations in the rhythmic human activities. In states *B* and *C* the dynamics is deterministic and corresponds to the relaxation and firing of the neurons. *State B* represents the “waiting time” or the rhythm giving part of the cycle. In biological systems this is a period the individual units want to impose, and usually this is the longest part of the cycle. The length of state *B* (τ_{B_I} or $\tau_{B_{II}}$) distinguishes the two modes. We have chosen $\tau_{B_I} = 2\tau_{B_{II}}$. The output of the units is in *state C*. During this state the oscillator emits a constant intensity pulse of strength $\frac{1}{N}$, where *N* is the number of oscillators in the system. The output of the whole system at a given moment is thus

$$f = \sum_{i=1}^{N} f_i \, , \quad (2)$$

where f_i is the output of oscillator *i:* $f_i = \frac{1}{N}$ if the given oscillator is in state *C*, and $f_i = 0$ otherwise.

This total output is the origin of the coupling and shifts the oscillators between their operating modes. The rules for the evolution of the system are as follows: (i) oscillators start with randomly selected modes and phases and follow the stated dynamics; (ii) there is a fixed output intensity, f^*, for the system; (iii) after completing the dynamics in state *A*, each oscillator will choose to operate either in mode I or mode II; (iv) if at that moment $f < f^*$ the oscillator will operate in mode I; otherwise it will follow mode II. The above dynamics has the tendency to keep the average total output as close as possible to f. Since each oscillator has a fixed output intensity, this can be achieved sometimes only by switching between the available modes. In this sense the proposed rules are natural, making our model realistic.

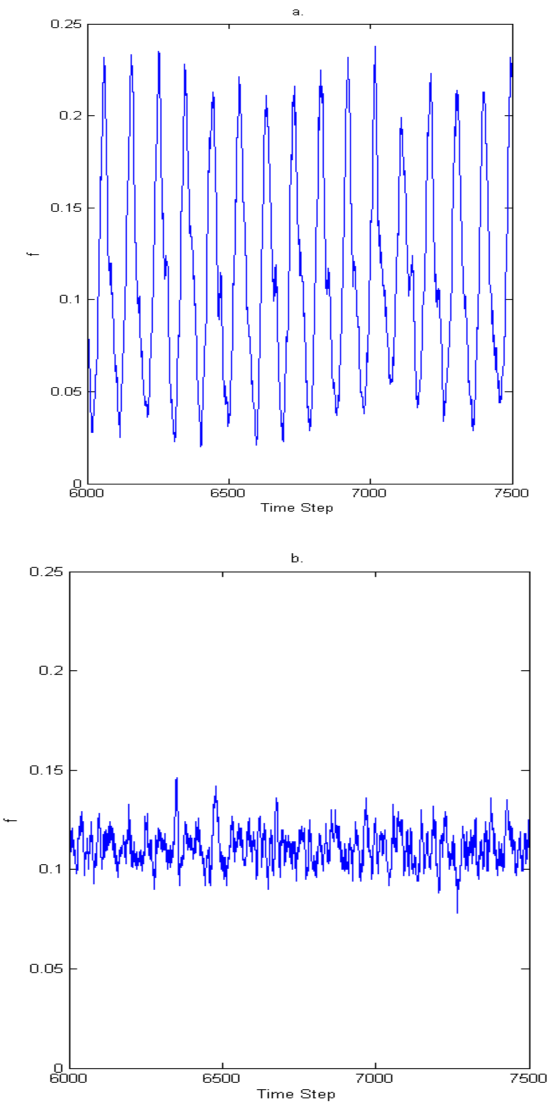


Fig. 1. Global output of the system for two different f values. The first case corresponds to a synchronized phase and the second one to an unsynchronized behavior. Number of oscillators in the simulation is: 1000, in case a. $f^* = 0.15$; *for b.* $f^* = 0.4$

For given τ^* and f^* values a synchronization-like phenomenon appears in the system. Synchronization in this system means that the global output has a periodic nature due to the fact the a large part of the oscillators produces output signal at relatively close time moments.

This is presented on Fig. 1. In figure 1.a. oscillators work almost synchronously, and it is evident that the average output is around the desired $f^* = 0.15$ value. In Fig. 1b there is no synchronization and the desired output level ($f^* = 0.4$) is not reached.

A more complete analysis of this synchronization will be given in the next chapter.

It is important to emphasize again that within this model the only goal of the oscillators is to hold the global output level around a fixed f^* value. There is no direct driven force favoring synchronization, and synchronization in this model is a highly nontrivial secondary effect.

The aim of the present work is to generalize the two-mode oscillator model to the n-mode case and to investigate what happens with the observed nontrivial synchronization if the number of modes are increasing.

3. The n-mode Stochastic Oscillator model

By increasing the number of modes two important questions arises: (i) whether synchronization gets better or worst, (ii) whether it is present for a larger or smaller f interval. These questions will be answered by performing stochastic simulations.

Three, five and nine-mode stochastic oscillators will be considered. States A and C will remain the same as in the two-mode case. In the three-mode case there are three τ_B values chosen as: $\tau_{B_I} = 0.4, \tau_{B_{II}} = 0.6, \tau_{B_{III}} = 0.8$. In the five- mode case we have chosen: $\tau_{B_I} = 0.4$, $\tau_{B_{II}} = 0.5$, $\tau_{B_{III}} = 0.6$, $\tau_{B_{IV}} = 0.7$, $\tau_{B_V} = 0.8$; and in the nine-mode case: $\tau_{B_I} = 0.4$, $\tau_{B_{II}} = 0.45$, $\tau_{B_{III}} = 0.5$, $\tau_{B_{IV}} = 0.55$, $\tau_{B_V} = 0.6$, $\tau_{B_{VI}} = 0.65$, $\tau_{B_{VII}} = 0.7$, $\tau_{B_{VIII}} = 0.75$, $\tau_{B_{IX}} = 0.8$.

As a first hint, synchronization can be examined from the Fourier transform of the output signal. Peaks will indicate the periodicity of this. As an example in such sense on Figure 2. results for the five-mode oscillators are plotted considering four different f threshold values. The Fourier transform suggest partially synchronized states for two of them ($f^* = 0.10$ and $f^* = 0.15$). It is easy to show that the n-mode case is similar with the two-mode oscillator system. As f increases the systems collective output will change from an unsynchronized phase to a synchronized one, and finally back again to an unsynchronized phase. The series given in Figure 2 illustrates this trend.

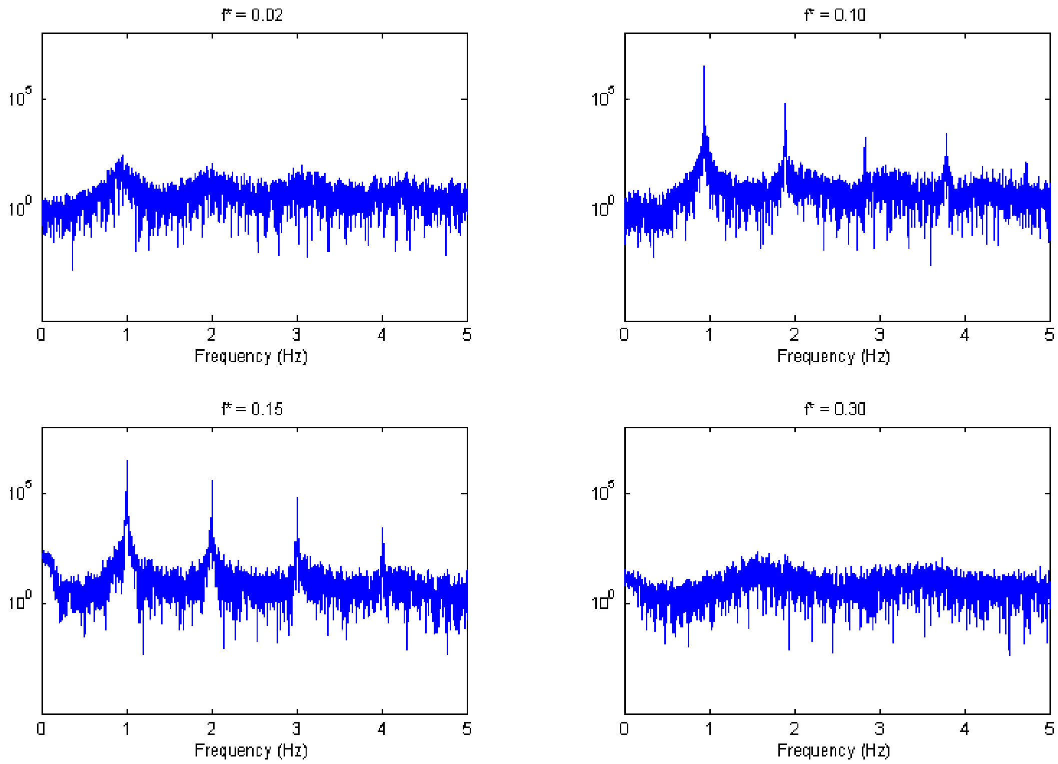


Fig. 2. The Fourier transform of the global output (N = 5000, $\tau^* = 0.2$, 5 modes)

4. Phase-space of the n-mode oscillator ensemble

For fixed τ_B and τ_C values the parameters governing the system dynamics are f^* and τ^*. Following the method used for two-mode oscillators [12], we would like now to map the whole $\{f^*, \tau^*\}$ phase-space and calculate the total output signal periodicity level in order to recognize the synchronized phases. For this a numerically computable periodicity measure is defined as follows.

Let us denote the output signal as a function of time as $f(t)$. We can define an error function, $\Delta(T)$, which characterizes numerically how strongly the $f(t)$ signal differs from a periodic signal with period T,

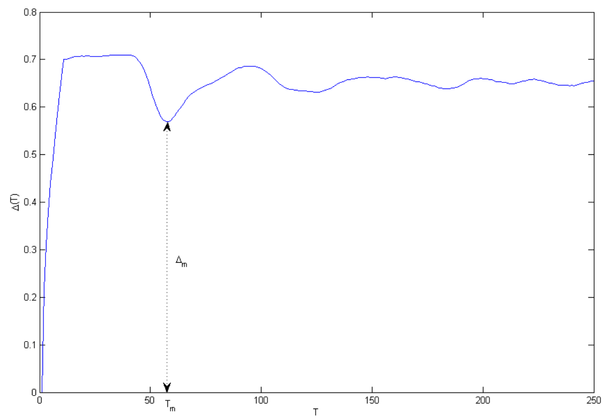
$$\Delta(T) = \frac{1}{2M}\lim_{x\to\infty}\frac{1}{x}\int_0^x \left|f(t) - f(t+T)\right|dt \qquad (3)$$

where

$$M = \lim_{x\to\infty}\frac{1}{x}\int_0^x \left|f(t) - \langle f(t)\rangle\right|dt \qquad (4)$$

$$\langle f(t)\rangle = \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt \qquad (5)$$

The general shape of the $\Delta(T)$ curve as a function of T is sketched on Fig. 3.



$\tau = 0.2$

Fig. 3: Plot of the $\Delta(T)$ error function.

For any $f(t)$ oscillating function we have an initially increasing tendency at small T values, after which for $T = T_m$ a minimum (Δ_m) is reached. One can state that T_m is the best approximation for the $f(t)$ signals period, and the "periodicity level" of the signal is characterized by

$$p = \frac{1}{\Delta_m} \qquad (6)$$

It is possible to compute this measure both for one oscillator working independently ($p1$) in the long period mode (where the effect of randomness on the period is smaller) and for the whole system (p). The ratio $p/p1$ will characterize then the enhancement in the periodicity due to the considered coupling. This ratio will also indicate the synchronization level of the oscillator ensemble. We have analyzed throughout the $\{f^*, \tau^*\}$ parameter space this measure. Results for $N = 1000$ oscillators are given in Fig. 4.

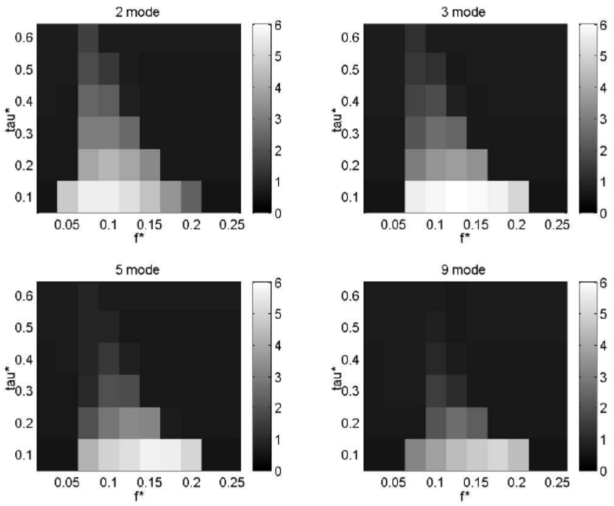


Fig. 4 Synchronization level (p/p1) illustrated in a gray-scale code in the $\{f^*, \tau^*\}$ parameter space. ($N = 1000$ oscillators). Lighter colors means stronger synchronization. The gray-scale code is given on the left-side of the images.

As expected, in the $\{f^*, \tau^*\}$ parameter space there is an island-like structure where synchronization is present. The brighter is a point in this parameter space the stronger is the synchronization. One can immediately observe that increasing the number of modes the size of the synchronized island diminishes. Another observation is that with the number of modes the synchronization becomes weaker. Naturally, synchronization is better for smaller values of τ^* where the effect of the randomness is smaller. For $\tau > 0.6$, synchronization completely vanishes. Concerning the dependence as a function of f^* we conclude that synchronization is present only for a limited interval. Fig. 4 suggest the same picture as Figure 2, i.e. by increasing the value f^* of the systems' collective output changes from the unsynchronized behavior to the synchronized dynamics, leading finally an unsynchronized regime again.

For Figure 4, simulations were performed with the same N=1000 oscillator number. Now let us study and compare the p/p1 values for different oscillator numbers as well. For fixed τ^* and f^* but different oscillator numbers the p/p1 periodicity level is plotted on Figure 5. Similarly with the two-mode oscillator case one gets that by increasing the number of oscillators, the periodicity is enhanced. Fig. 5 suggests again that by increasing the number of modes synchronization gets worst.

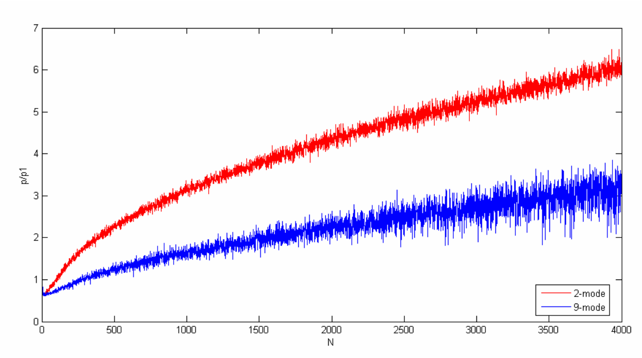


Fig. 5. Enhancement in the periodicity of the output (p/p1) as a function of the number of oscillators for the two- and nine-mode oscillator system. Upper curve is for the two-mode oscillators, and the curve from below is for the nine-mode oscillators ($\tau^* = 0.2$, $f^* = 0.15$ for both cases).

5. Phase-transition and mode-distribution in the n-mode oscillator system

Let us study now the emergence and disappearance of the periodic behavior (synchronization) in the n-mode oscillator system as f^* is varied. If we consider the order parameter as (p/p1), the variation of this quantity as a function of f^* suggests two first-order phase transitions: one for the appearance of the synchronized phase and one for its disappearance. Results for various number of modes and system sizes are plotted in Figure 6. The scaling with system size and the abrupt trend of this variation are all in agreement with the presumed first-order nature of this transition. The results also suggest that the interval where synchronization is present gets smaller as the number of modes increases. The f^* values where the synchronization appears and dissapears can be considered as critical values.

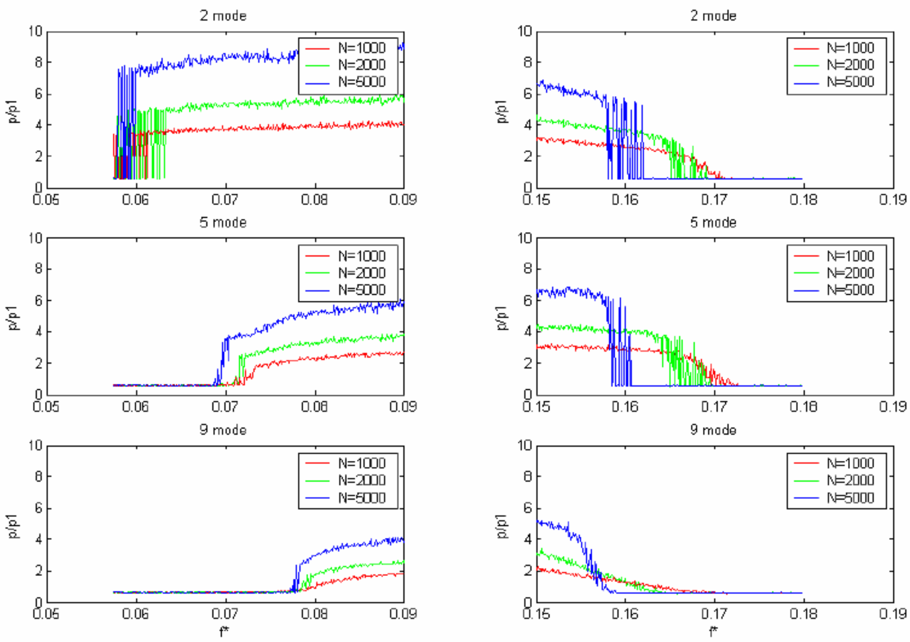


Fig. 6. First-order phase transitions for the emergence and disappearance of the synchronized phase (order parameter as a function of f^*, for $\tau^* = 0.2$.

It is also instructive to see the distribution of the modes in which the oscillators operate. In other words this means that we construct the histograms showing the percentage of the oscillators working in different modes, averaged for a long dynamics. It is evident that if f^* is small, all oscillators will follow the mode which corresponds to the longest period. If f^* is big, all oscillator will choose the smallest period. On the other hand, in the synchronized phase all modes are present. Results in this sense are presented on Figure 7. Moreover, most of the oscillators will not keep their mode during the dynamics and will continuously shift between the available modes. Synchronization will appear as a result of this.

6. Conclusions

A multi-mode stochastic oscillator system coupled through their pulse-like output was studied by Monte Carlo simulations. The oscillators are realistic in the sense that they model the behavior of several biological and sociological systems, where the individuals are capable of imposing various discrete periods and tend to optimize a global output. In our case the system is designed to keep a given global output level, and no direct interaction favoring synchronization is present. As a highly nontrivial effect it was found that the system synchronizes for a given interval of the desired global output. In this synchronized regime the periodicity level of the global output can exceed the periodicity level of one stochastic oscillator. The transition between the synchronized and non-synchronized regimes shows evidences of a first-order phase transition. The best synchronization (highest periodicity level of the global output) was achieved for two-mode oscillators. Increasing the number of modes will not favor synchronization. The periodicity level of the global output will decrease, and also the parameter region where partial synchronization is present will decrease as the number of modes are increased.

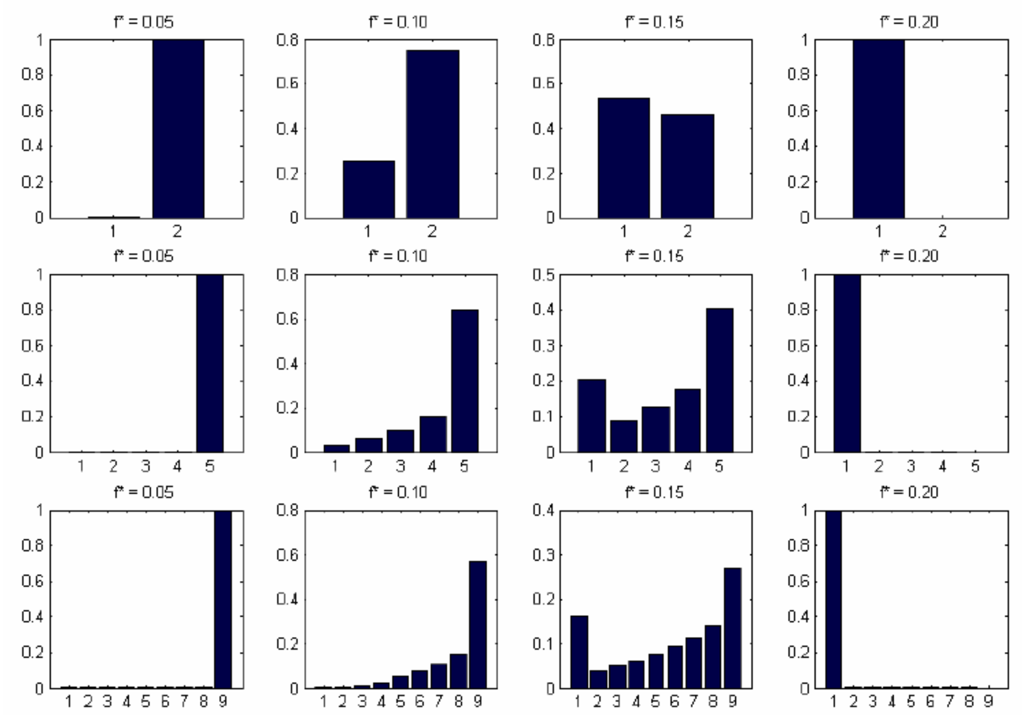


Fig. 7. Distribution of the modes in which the oscillators operate. (N = 1000, $\tau^* = 0.2$)

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