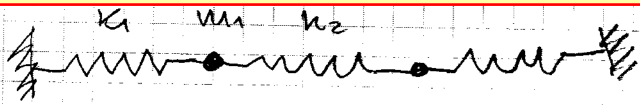
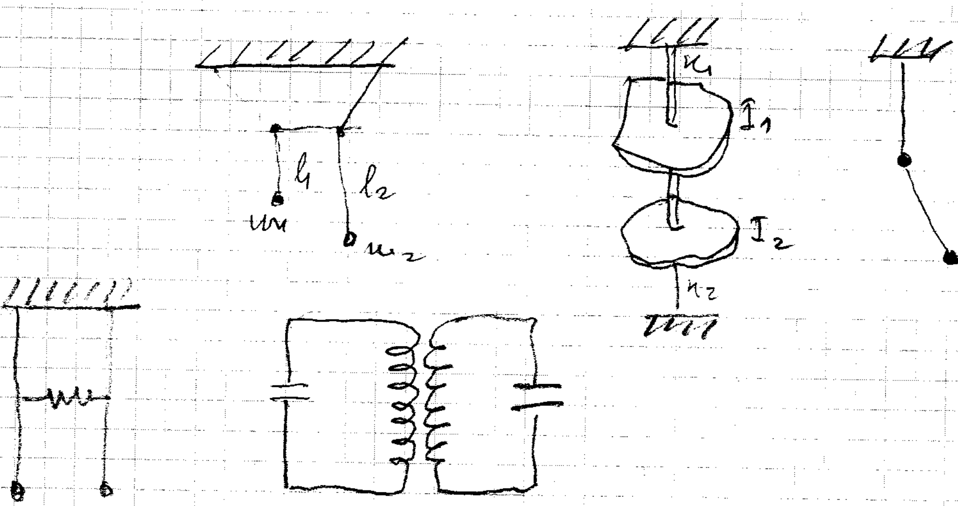
Oscilatori cuplați.
Oscilațiile sistemelor cu mai mult grade de libertate

ex.





ex   2 grade de libertate / 2 variabile

- mișcare este complicată și în general nu este armonică

- pt ecuații liniare → superpoziția a două mișcări simple armonice care au loc simultan

moduri normale

condițiile inițiale → un mod

Proprietățile unui mod DATUM:

- dacă este prezent doar un mod fiecare parte mobilă execută o mișc. armonică.

modul 1
$$x_1(t) = A_1 \cos(\omega_1 t + \varphi_1)$$
$$x_2(t) = B_1 \cos(\omega_1 t + \varphi_1) = \frac{B_1}{A_1} x_1(t)$$

modul 2
$$x_1(t) = A_2 \cos(\omega_2 t + \varphi_2)$$
$$x_2(t) = B_2 \cos(\omega_2 t + \varphi_2) = \frac{B_2}{A_2} x_1(t)$$

fiecare mod → frecvența caracteristică

→ configurație caract.

pt. 1 A_1/B_1, pt. 2 A_2/B_2

$$\frac{x_1(t)}{x_2(t)} = \text{const.}$$

mișcarea cea mai generală

⇕

$$x_1(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2)$$
$$x_2(t) = B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2)$$

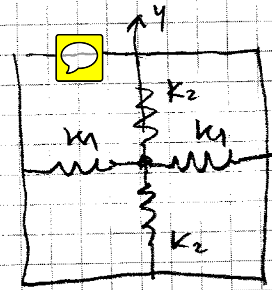
ex. pendulul sferic simplu

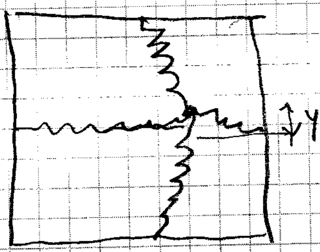
pe x și y aceeași frecvență $\omega^2 = g/l$

degenerate $\left| \begin{array}{l} x(t) = A_1 \cos(\omega_1 t + \varphi_1) ;\ \omega_1 = \omega \\ y(t) = B_2 \cos(\omega_2 t + \varphi_2) ,\ \omega_1 = \omega_2 = \omega \end{array} \right.$

Coord normale

DATUM:





neglijăm x^2, y^2, xy

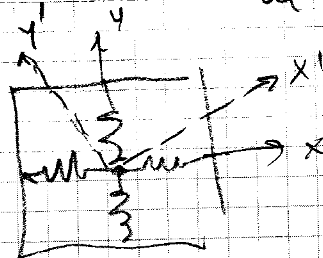
$$m\frac{d^2x}{dt^2} = -2k_1 x \quad , \quad m\frac{d^2y}{dt^2} = -2k_2 y$$

$$x = A_1 \cos(\omega_1 t + \varphi_1) \qquad \omega_1^2 = \frac{2k_1}{m}$$

$$y = B_2 \cos(\omega_2 t + \varphi_2) \qquad \omega_2^2 = \frac{2k_2}{m}$$

x, y − coord. normale

ecuații decuplate



dacă am folosi alte coordonate x' si y'

x, y − comb. liniară de x' și y'

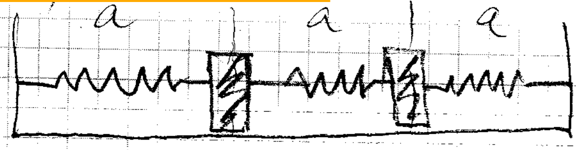
ecuații cuplate

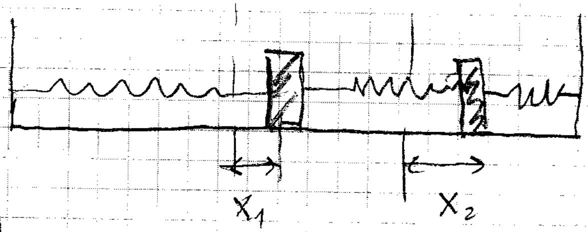
a.

$$\frac{d^2x}{dt^2} = -a_{11}x - a_{12}y$$

$$\frac{d^2y}{dt^2} = -a_{21}x - a_{22}y$$

Oscilatori cuplati identici

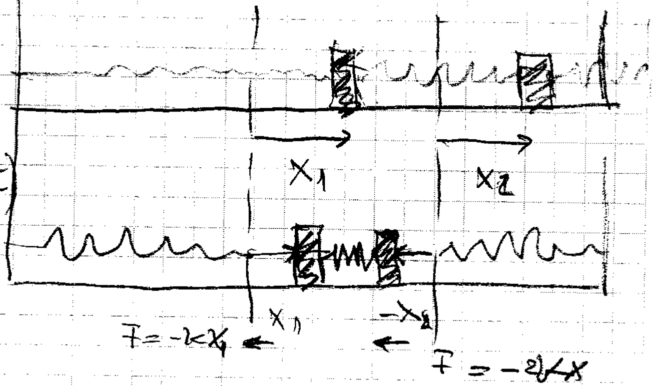




modurile norma

1. $x_1(t) = x_2(t)$

$$\omega_1^2 = \frac{k}{m}$$



2. $x_1(t) = -x_2(t)$

$$\omega_2^2 = \frac{3k}{m}$$

în general

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

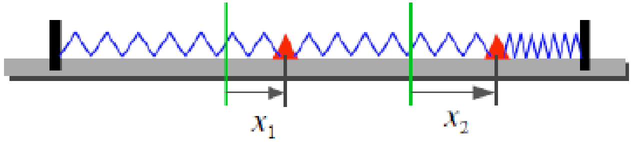
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$

ecuaii cupl

$$\begin{cases} \ddot{x}_1 + \frac{k}{m}(2x_1 - x_2) = 0 \\ \ddot{x}_2 + \frac{k}{m}(2x_2 - x_1) = 0 \end{cases}$$

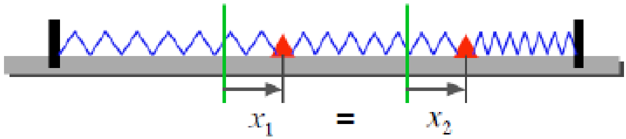
cutm coord

dac adunm i scdem cele dou ecuaii, prin alegerea conv



exemplu c

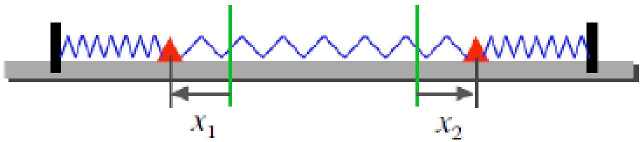
1.



$$\omega_0 = \sqrt{\frac{k}{m}} \, .$$

modul fundamental

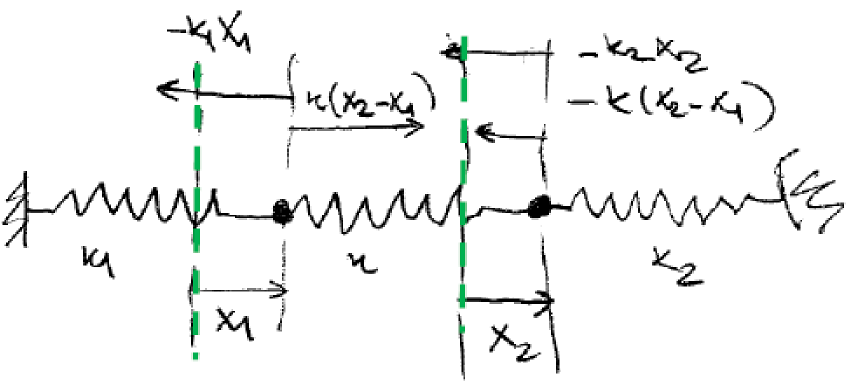
2.

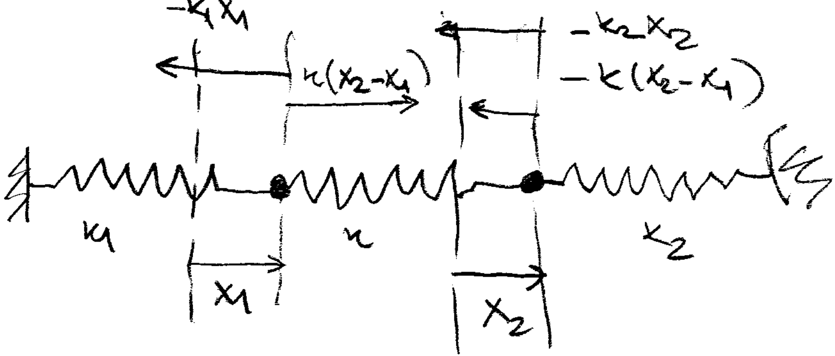


$$\omega = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_0 \, .$$

modul excitat

in general:





$-k_1 x_1$
$k(x_2 - x_1)$
$-k_2 x_2$
$-k(x_2 - x_1)$
k_1
k
k_2
x_1
x_2

DATUM:

$$\boxed{\begin{aligned}&(\ddot{x}_1+\ddot{x}_2)+\frac{k}{m}(x_1+x_2)=0\\&(\ddot{x}_1-\ddot{x}_2)+3\frac{k}{m}(x_1-x_2)=0\end{aligned}}$$

ecuaii decupla

$$\begin{array}{ll} X_+ & \left\{ \begin{array}{l} x_1+x_2=A_1\cos(\omega t+\varphi_1) \\ x_1-x_2=A_2\cos(\sqrt{3}\,\omega t-\varphi_2) \end{array}\right. \\ X_- & \end{array} \qquad \omega^2=\frac{k}{m}$$

coordonatele norm

$\frac{1}{2}X_+$ — mișcarea centrului de masă

X_- - deplasarea relativă

dac adunm/scdem cele dou soluii gsi

$$2x_1(t)=A_1\cos(\omega_1t+\varphi_1)+A_2\cos(\omega_2t+\varphi_2)$$

$$2x_2(t)=A_1\cos(\omega_1t+\varphi_1)-A_2\cos(\omega_2t+\varphi_2)$$

modul 1 $\quad A_2=0 \Rightarrow x_1(t)=x_2(t)$

2 $\quad A_1=0 \quad x_2(t)=-x_1(t)$

Putem selecta un anumit mod normal de vibraie, alegând

Valori proprii, vectori proprii (un alt li

$$m\ddot{x}_1 = -2kx_1 + kx_2$$

$$m\ddot{x}_2 = kx - 2kx_2$$

acesta este sistemul de ecuaii

sub forma matrici

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = -\frac{k}{m}\underbrace{\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\lambda = \left(\frac{\omega}{\omega_0}\right)^2$$

vezi (

ecuaia cu valori propri:

$$\det(\lambda \mathbb{1} - M) = 0 \qquad \mathbb{1} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{vmatrix}\lambda-2 & 1\\ 1 & \lambda-2\end{vmatrix} = 0 \qquad (\lambda-2)^2 - 1 = 0$$

$$\lambda = 3 \qquad \lambda = 1$$

$$\omega_1 = \sqrt{3\frac{k}{m}}$$

$$\omega^2 = \lambda\frac{k}{m}$$

$$\omega_2 = \sqrt{\frac{k}{m}}$$

să găsim un vector propriu pt. $\lambda = 3$

$$\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 3\begin{bmatrix}a\\ b\end{bmatrix}$$

$$2a - b = 3a \qquad a + b = 0 \qquad a = -b$$

$$-a + 2b = 3b \qquad \begin{bmatrix}a\\ b\end{bmatrix} \sim \begin{bmatrix}1\\ -1\end{bmatrix}$$

componentele sunt op

(* $U = A.V$

$$U = \underline{AV = \lambda V}$$

ecuaia cu val

$$[A - \lambda \mathbb{1}][V] = 0$$

componentele sunt ~~opuse~~ x_1/x_2

pt $\omega = \sqrt{3}\,\omega_0$

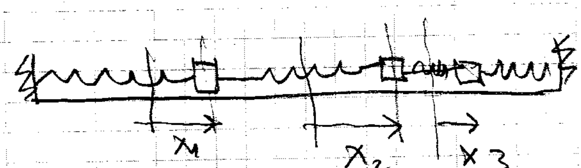
dacă $\lambda = 1$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_1 = x_2$$

cazul a trei corpuri

un ex util

să arătați



$$\ddot{x}_1 = -\frac{k}{m}(2x_1 - x_2)$$

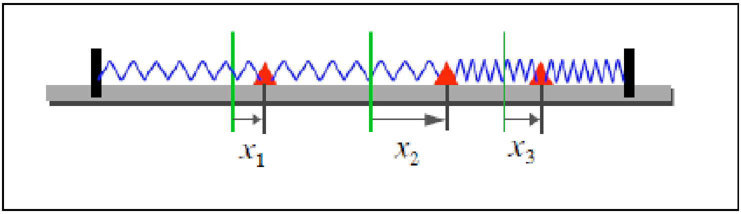
$$\ddot{x}_2 = -\frac{k}{m}(-x_1 + 2x_2 - x_3)$$

$$\ddot{x}_3 = -\frac{k}{m}(-x_2 + 2x_3)$$

trebuie să găsim valorile proprii și vect. p.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Cazul a trei corpuri



$$\ddot{x}_1 = -\frac{k}{m}(2x_1 - x_2 \qquad)$$

$$\ddot{x}_2 = -\frac{k}{m}(-x_1 + 2x_2 - x_3)$$

$$\ddot{x}_3 = -\frac{k}{m}(\qquad -x_2 + 2x_3)$$

matricea sistemului

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

ecuatia caracteristica

$$\det\begin{bmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{bmatrix} = (\lambda-2)\left((\lambda-2)^2-1\right)-(\lambda-2)$$

$$=(\lambda-2)\left((\lambda-2)^2-2\right)$$

$$=0$$

solutiile

$$\boxed{\lambda = 2-\sqrt{2},\ 2,\ 2+\sqrt{2}}$$

vectorii proprii:

$$\boxed{\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}}$$

DATUM:

$$\begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix} = (\lambda-2)\left((\lambda-2)^2-1\right)-(\lambda-2)$$

$$= (\lambda-2)\left((\lambda-2)^2-2\right) = 0$$

$\lambda = 2-\sqrt{2}\ ,\ 2\ ,\ 2+\sqrt{2}$

$\omega_i = \sqrt{\lambda_i}\ \omega_0$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$2a - b = 2a \rightarrow b = 0$

$-a + 2b - c = 2b \rightarrow a = -c$

$-b + 2c = 2c$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

corpul din mijloc

stă pe loc

$\frac{1}{\sqrt{2}}$ $\quad$ $\frac{1}{\sqrt{2}}$

1 $\quad$ -1

$\frac{1}{\sqrt{2}}$ $\quad$ $\frac{1}{\sqrt{2}}$

generalizare → matricea $n \times n$, 2 pe diag. și −1 de-o parte și alta și zero altele

→ are n valori proprii λ_i și n vectori proprii

Frecv. mod normale $\omega_i = \sqrt{\lambda_i}\ \omega_0$

Generalizare

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \\ \ddot{z}_N \end{bmatrix} = -\omega_0^2 \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ & \cdots & & & \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = (A\cos\omega t + B\sin\omega t) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \frac{\omega^2}{\omega_0^2} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

$$\lambda$$

$$\lambda_n = 2 - 2\cos\left(\frac{n\pi}{N+1}\right)$$

$$v^n = \begin{pmatrix} \sin\frac{n\pi}{N+1} \\ \sin\frac{2n\pi}{N+1} \\ \vdots \\ \sin\left(\frac{Nn\pi}{N+1}\right) \end{pmatrix}$$

soluțiile

$$\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{v}_i \cos \omega_i t$$

am luat vitezele inițiale zero

---

mișcarea lanțului nu e neapărat periodică — suma unor mișcări simple

moduri normale

vectorii proprii ortogonali

$$\vec{v}_i \vec{v}_j = 0 \qquad \vec{v}_i \vec{v}_i = 1$$

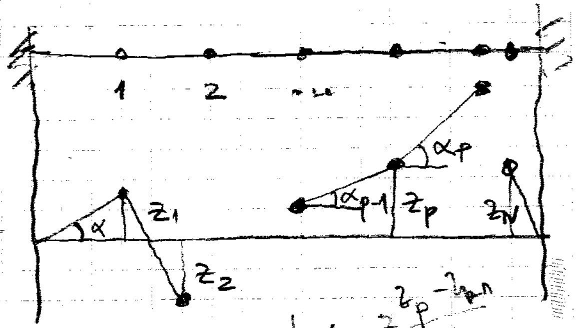
$$\vec{x}(0) = \sum_{i=1}^{n} c_i \vec{v}_i \rightarrow \vec{v}_j \cdot \vec{x}(0) = \sum c_i (\vec{v}_j \vec{v}_i)$$

$$\vec{v}_j \cdot \vec{x}(0) = c_j$$

dacă este pornit într-un mod normal k

$$c_k \neq 0 \rightarrow \vec{x}(t) = c_k \vec{v}_k \cos \omega_k t$$

Extindere



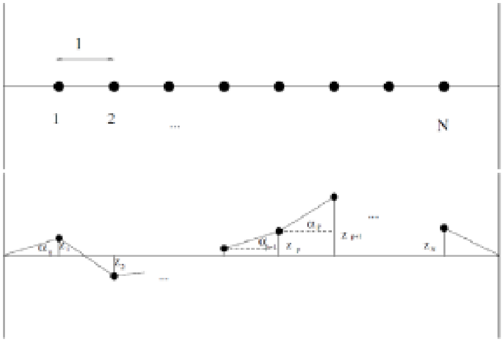
$\sin\alpha_p \approx \operatorname{tg}\alpha_p \approx \alpha_p$, $\cos\alpha_p \approx 1$

$T_p^o = -T\cos\alpha_{p-1} + T\cos\alpha_p \approx 0$ orr

$T_p^v = -T\sin\alpha_{p-1} + T\sin\alpha_p$

$\operatorname{tg}\alpha_{p-1} = \frac{z_p - z_{p-1}}{l}$

$$F_p^v = -\frac{\tau}{\ell}\left(z_p - z_{p-1}\right) + \frac{\tau}{\ell}\left(z_{p+1} - z_p\right) = m\ddot{z}$$



un sir de corpuri

$$\boxed{\begin{gathered}\sin\alpha_p \approx \tan\alpha_p \approx \alpha_p, \quad p = 1, \ldots, N \\ \\ \cos\alpha_p \approx 1, \quad p = 1, \ldots, N.\end{gathered}}$$

`pentru unghiuri`

$$F_p^{\text{horizontal}} = -\tau\cos\alpha_{p-1} + \tau\cos\alpha_p.$$

$$\boxed{F_p^{\text{horizontal}} \approx 0.}$$

$$F_p^{\text{vertical}} = -\tau\sin\alpha_{p-1} + \tau\sin\alpha_p$$

$$\boxed{\tan\alpha_{p-1} = \frac{z_p - z_{p-1}}{l}}$$

$$F_p^{\text{vertical}} = -\frac{\tau}{l}(z_p - z_{p-1}) + \frac{\tau}{l}(z_{p+1} - z_p)$$

p merge de la 1 la *N*.

$$m\ddot{z}_p = -\frac{\tau}{l}(z_p - z_{p-1}) - \frac{\tau}{l}(z_p - z_{p+1})$$

pentru primul si ultimul corp:

$$\begin{aligned} m\ddot{z}_1 &= -\frac{\tau}{l}z_1 - \frac{\tau}{l}(z_1 - z_2) \\ \ddot{z}_N &= -\frac{\tau}{l}(z_N - z_{N-1}) - \frac{\tau}{l}z_N \end{aligned}$$

gasim aceleasi ecuatii ca in cazul sistemelor bloc+resort

$\varphi_1 = \varphi_2 = 0$ DATUM: $A_1 = A_2$

Revenim

$$x_1 = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$$

$$= A_1 (\cos \omega_1 t + \cos \omega_2 t) =$$

$$= \left[2 A_1 \cos \tfrac{1}{2}(\omega_1 - \omega_2) t \right] \cos \tfrac{1}{2}(\omega_1 + \omega_2) t$$

$$\left(2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta) \right)$$

$$x_2 = A_1 \cos \omega_1 t_1 - A_1 \cos \omega_2 t$$

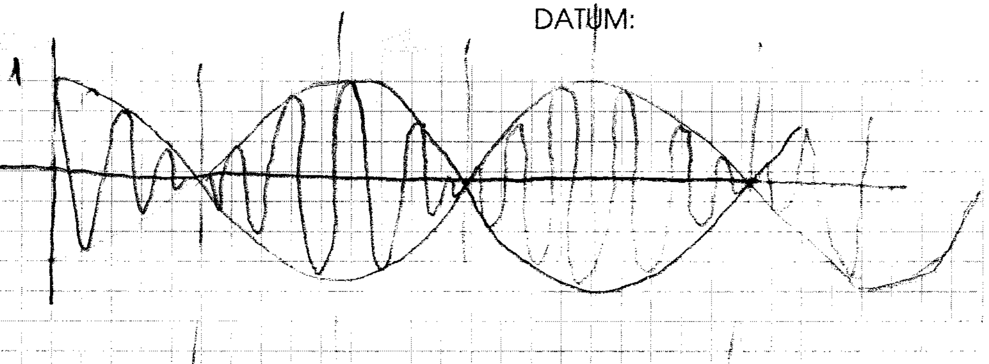
$$= \left[2 A_1 \sin \tfrac{1}{2}(\omega_1 - \omega_2) t \right] \sin \tfrac{1}{2}(\omega_1 + \omega_2) t$$

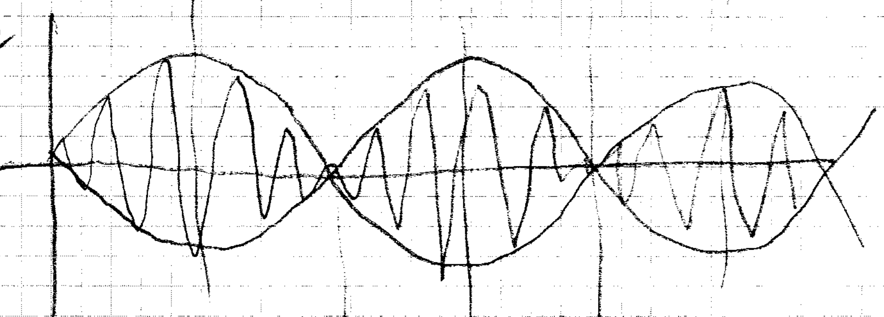
amplitudinea modulată pt x_1

$$2 A_1 \cos \tfrac{1}{2}(\omega_1 - \omega_2) t$$

pt x_2

$$2 A_1 \sin \frac{1}{2} (\omega_1 - \omega_2) t = 2A \cos \left[\frac{1}{2} (\omega_1 - \omega_2) t - \frac{\pi}{2} \right]$$





transfer de energie

Energia totală $\quad E = E_c + E_p$

$$E_c = \frac{1}{2} m v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$E_p = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$$

$$E = \overbrace{\left[\frac{1}{2} m_1 v_1^2 + \frac{1}{2}(k_1 + k) x_1^2\right]}^{m_1} + \overbrace{\left[\frac{1}{2} m_2 v_1^2 + \frac{1}{2}(k_2 + k) x_2^2\right]}^{m_2}$$

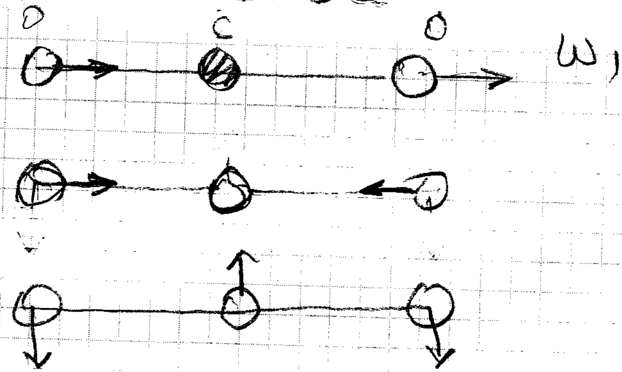
$- k x_1 x_2$ cuplaj (în absența lui energ 1 și energ 2 = const)

în general

$$E = (E_c + E_p)_1 + (E_c + E_p)_2 + (E_p)_{12} = \text{const.}$$

ex. atomii în molecule

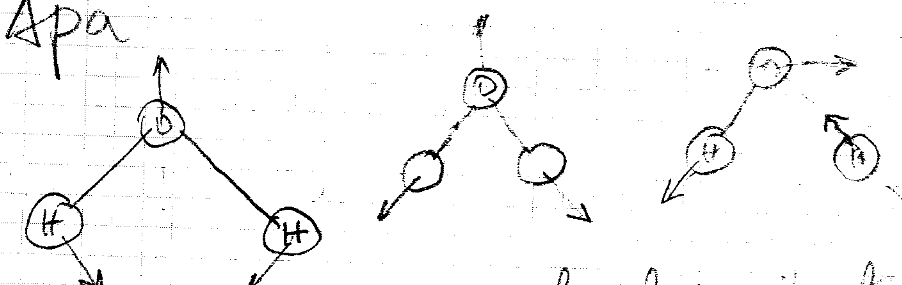
CO_2



$\omega_1 = 4.443 \cdot 10^{14} s^{-1}$, $\omega_2 = 2.529 \cdot 10^{14} s^{-1}$

$\omega_3 = 1.261 \cdot 10^{14} s^{-1}$

Apa

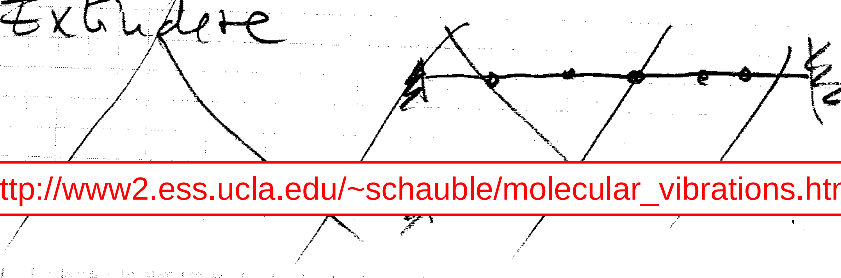


molecular vibration

$\measuredangle\ 105^\circ$

$\omega_1 = 3 \cdot 10^{14} s^{-1}$ $\quad \omega_2 = 6{,}908 \cdot 10^{14} s^{-1}$ $\quad \omega_3 = 7{,}104 \cdot 10^{14}$

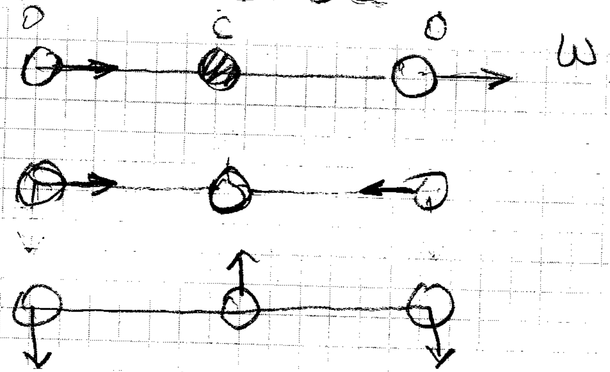
Extindere



http://www2.ess.ucla.edu/~schauble/molecular_vibrations.htm ex. atomii în molecule

DATUM:

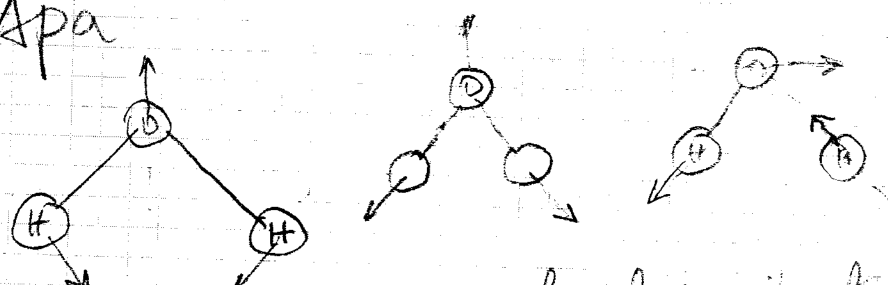
CO_2



$\omega_1 = 4.443 \cdot 10^{14}\, s^{-1}$, $\omega_2 = 2.529 \cdot 10^{14}\, s^{-1}$

$\omega_3 = 1.261 \cdot 10^{14}\, s^{-1}$

Apa



molecular vibration

$\measuredangle\ 105^\circ$

$\omega_1 = 3 \cdot 10^{14}\, s^{-1}$ $\omega_2 = 6.908 \cdot 10^{14}\, s^{-1}$ $\omega_3 = 7.164 \cdot 10^{14}$

Extindere