

## The problem of plotting Series

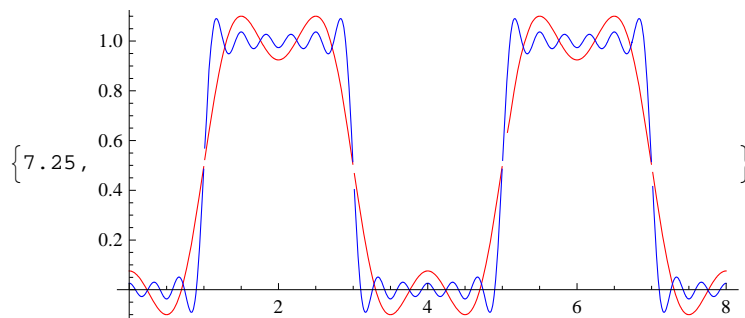
<http://cda.mrs.umn.edu/~mcquarrb/DigitalDE/Math/BesselFunctions.pdf>

We want to plot the expression  $\sum_{n=0}^m \frac{2(-1)^{n+1}}{(2n+1)\pi} \cos\left[\frac{(2n+1)\pi}{2} t\right]$

Our purpose is to plot the first m terms and for this reason is necessary to define

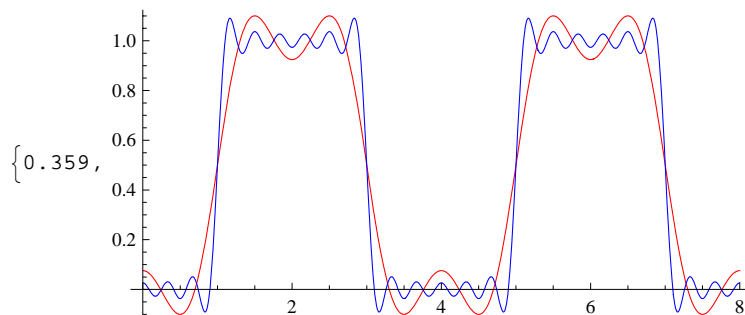
$$f[x_, m_] = \frac{1}{2} + \sum_{n=0}^m \frac{2(-1)^{n+1}}{(2n+1)\pi} \cos\left[\frac{(2n+1)\pi}{2} t\right];$$

```
Timing[Plot[{f[x, 1], f[x, 5]}, {t, 0, 8},
  AspectRatio -> 1/2, PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}]]
```



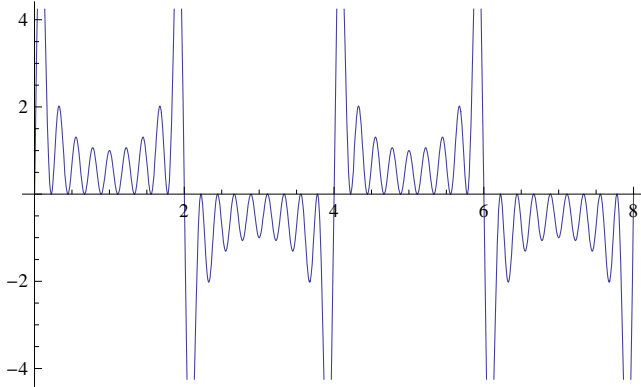
Defining g as above causes Mathematica to algebraically manipulate the series into known functions (which in this case is an exact summation formula in terms of confluent hypergeometric functions). During plotting these special functions must be evaluated, using up too much cpu time. In many cases it is much faster (but more cluttered) to just insert the series directly into the Plot function.

```
Timing[Plot[{1/2 + Sum[(2(-1)^(n+1)) Cos[1/2 ((2n+1) pi) t] / ((2n+1) pi), {n, 0, 5}], 1/2 + Sum[(2(-1)^(n+1)) Cos[1/2 ((2n+1) pi) t] / ((2n+1) pi), {n, 0, 5}], {t, 0, 8}, AspectRatio -> 1/2, PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}]]
```



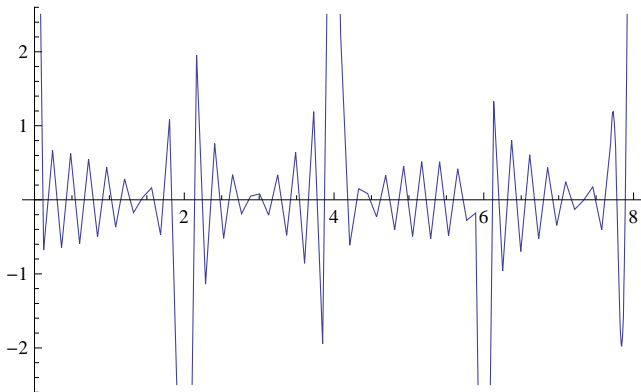
One nice convenience of using Mathematica is that most of the time it is able to choose the right number of grid points on which to plot.

```
g1 = Plot[Sum[Sin[1/2 ((2 n + 1) π) t], {n, 0, 8}], {t, 0, 8}]
```

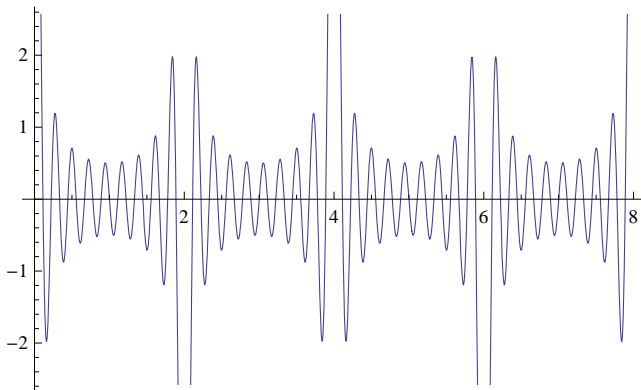


If you want to take control you can ask for more or less resolution using the PlotPoints option

```
g2 = Plot[Sum[Cos[1/2 ((2 n + 1) π) t], {n, 0, 8}], {t, 0, 8}, PlotPoints -> 2]
```

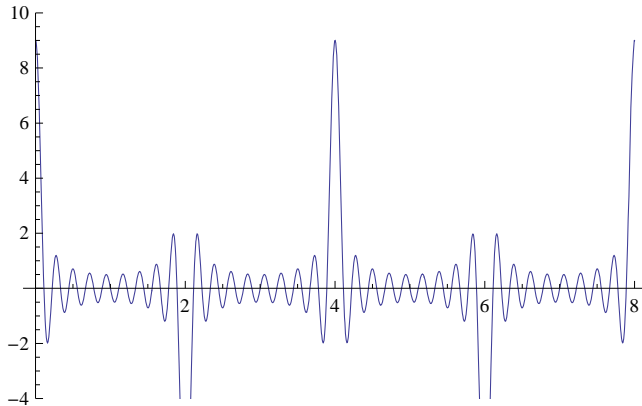


```
g3 = Plot[Sum[Cos[1/2 ((2 n + 1) π) t], {n, 0, 8}], {t, 0, 8}, PlotPoints -> 100]
```

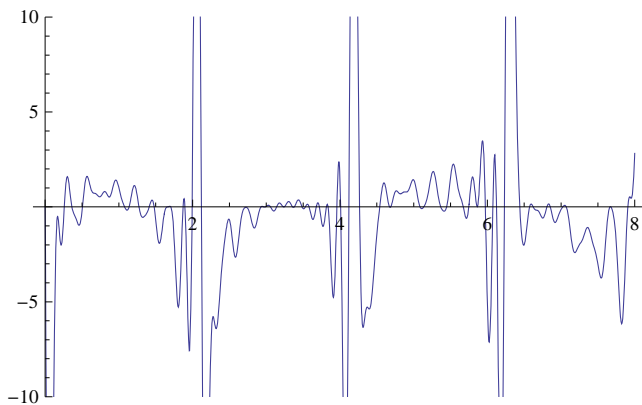


Mathematica also makes its best attempt to choose an optimal range over which to plot. If you feel that it is truncating your plots or not zooming in on the interesting parts, you can use the `PlotRange` option to take control.

```
g4 = Plot[Sum[Cos[1/2 ((2 n + 1) π) t], {n, 0, 8}], {t, 0, 8}, PlotRange -> {-4, 10}]
```



```
h1 = Plot[Sum[Cos[1/(2 * n + 2) ((2 n + 1 - 2 * n^2) π) t] * Sum[Sin[1/(2 * n + 2) ((2 n + 1 - 2 * n^2) π) t], {n, 0, 8}], {t, 0, 8}, PlotRange -> {-10, 10}]
```




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## Bessel's Equation

The Bessel's equation is:

$$\mathbf{L}[y\_ ] := \mathbf{x}^2 \mathbf{D}[y, \{x, 2\}] + \mathbf{x} * \mathbf{D}[y, \{x, 1\}] + (\mathbf{x}^2 - \mathbf{v}^2) y = 0$$

$$\mathbf{L}[y[x]]$$

$$(-v^2 + x^2) y[x] + x y'[x] + x^2 y''[x] = 0$$

*Mathematica* can solve this, and we get:

```
DSolve[L[y[x]], y[x], x]
{{y[x] -> BesselJ[v, x] C[1] + BesselY[v, x] C[2]}}
```

Our goal is to understand what how these two solutions ( Bessel Functions of the First Kind of order  $\nu$  ( $J_\nu(x)$ ) and Bessel Functions of the Second Kind of order  $\nu$  ( $Y_\nu(x)$ ) are obtained.

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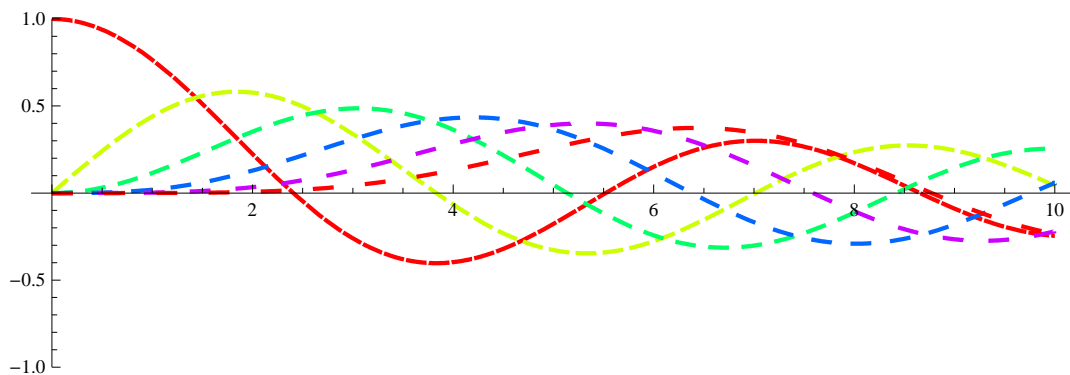
## Bessel Functions of The First Kind $J_\nu(x)$

$$J_{1\nu}[x] = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma[1 + \nu + m]}$$

```
BesselJ[v, x]
```

The sketch of  $J_\nu(x)$  for  $\nu=0,1,2,3,4,5$  is

```
BesJlist = Table[J1v[x], {v, 0, 5}];
huelist = Table[{Hue[k / 5], Dashing[{0.02, k / 5 * 0.03]}, Thick}, {k, 0, 5}];
Plot[Evaluate[BesJlist], {x, 0, 10}, PlotStyle -> huelist,
ImageSize -> {500, Automatic}, PlotRange -> {Automatic, {-1, 1}}, AspectRatio -> 1 / 3]
```




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## Bessel Functions of The Second Kind $Y_\nu(x)$

We try to look at finding the first few terms in the second solution using reduction of order. We use the formula for the second solution

$$y_2(x) = \left( \int \frac{e^{-\int p(x) dx}}{(y_1(x))^2} dx \right) y_1(x)$$

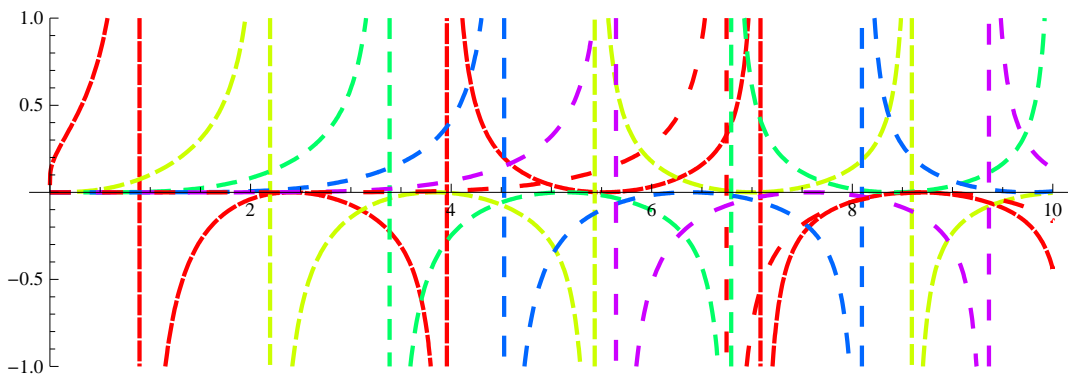
The most difficult aspect of the calculation is the squaring and inversion of a series, which is easy to do with a computer algebra system. This might not always be feasible! Here we are lucky in that we have the first solution as a Bessel function, so we do not have to work with the infinite series at all.

$$y_{1v}[x_] = \left( \int \frac{e^{-\int \frac{1}{x} dx}}{(y_{1v}[x])^2} dx \right) * J_{1v}[x]$$

$$\frac{2 \text{BesselJ}[v, x]^2}{\pi \text{BesselY}[v, x]}$$

Notice that we get the second solution as a Bessel Function of the Second Kind of order  $v$ , up to a constant multiple. To do this integral by hand we would need to use a Big Book of Integrals, or spend more time learning about Bessel functions.

```
BesYlist = Table[ $\frac{2}{\pi} y_{1v}[x]$ , {v, 0, 5}];
huelist = Table[{Hue[k / 5], Dashing[{0.02, k / 5 * 0.03}], Thick}, {k, 0, 5}];
Plot[Evaluate[BesYlist], {x, 0, 10}, PlotStyle -> huelist,
ImageSize -> {500, Automatic}, PlotRange -> {-1, 1}, AspectRatio -> 1 / 3]
```



## Bessel Functions of The Second Kind of Order 0, $Y_0(x)$ : Reduction of Order with Series Solution

Let's see what happens if we did not know our first solution was  $J_0(x)$ , and we truncated the series solution  $y_1(x)$  and worked out the second solution from that. We will truncate our first series at 10 terms.

$$J_{1t_{v,k}}[x_] := \left(\frac{x}{2}\right)^v \sum_{m=0}^k \frac{(-1)^m x^{2m}}{2^{2m} m! \text{Gamma}[1 + v + m]}$$

```

v1 = 0;
k1 = 10;

Jltv1,k1[x];
Expand[%]

```

$$1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \frac{x^{10}}{14745600} + \frac{x^{12}}{2123366400} - \frac{x^{14}}{416179814400} + \frac{x^{16}}{106542032486400} - \frac{x^{18}}{34519618525593600} + \frac{x^{20}}{13807847410237440000}$$

Now we square the series:

```

J1sq[x_] = Expand[Jltv1,k1[x]^2]

```

$$1 - \frac{x^2}{2} + \frac{3x^4}{32} - \frac{5x^6}{576} + \frac{35x^8}{73728} - \frac{7x^{10}}{409600} + \frac{77x^{12}}{176947200} - \frac{143x^{14}}{17340825600} + \frac{143x^{16}}{1183800360960} - \frac{2431x^{18}}{1725980926279680} + \frac{46189x^{20}}{345196185255936000} - \frac{583x^{22}}{5523138964094976000} + \frac{11173x^{24}}{1590664021659353088000} - \frac{127x^{26}}{31813280433187061760000} + \frac{487x^{28}}{24941611859618656419840000} - \frac{3691x^{30}}{4489490134731358155571200000} + \frac{6793x^{32}}{22986189489824553756524544000000} - \frac{41x^{34}}{45972378979649107513049088000000} + \frac{131x^{36}}{59580203157625243336911618048000000} - \frac{x^{38}}{238320812630500973347646472192000000} + \frac{x^{40}}{190656650104400778678117177753600000000}$$

All looks fine.

And invert the squared series:

```

inJ1sq[x_] = Normal[Series[1/J1sq[x], {x, 0, 2 * k1}]]

```

$$1 + \frac{x^2}{2} + \frac{5x^4}{32} + \frac{23x^6}{576} + \frac{677x^8}{73728} + \frac{7313x^{10}}{3686400} + \frac{218491x^{12}}{530841600} + \frac{863897x^{14}}{10404495360} + \frac{874088357x^{16}}{53271016243200} + \frac{27545803997x^{18}}{8629904631398400} + \frac{423385249313x^{20}}{690392370511872000}$$

Now, use the reduction of order formula to get the second solution:

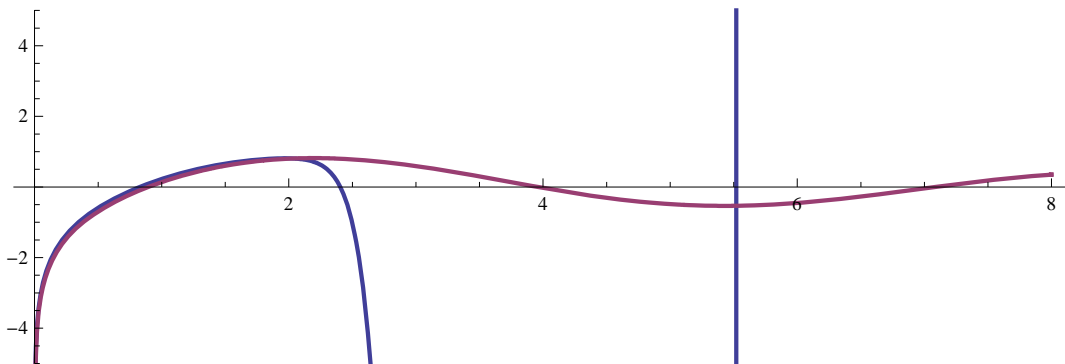
$$y2[x_] = \left( \int \text{inJ1sq}[x] * e^{-\int \frac{1}{x} dx} dx \right) * \text{J1t}_{v1,k1}[x];$$

`y2[x_] = Expand[y2[x]]`

$$\begin{aligned} & \frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13824} - \frac{25x^8}{1769472} + \frac{137x^{10}}{884736000} - \frac{49x^{12}}{42467328000} + \frac{121x^{14}}{19421724672000} - \\ & \frac{761x^{16}}{29831769096192000} + \frac{7129x^{18}}{86989438684495872000} - \frac{7381x^{20}}{3479577547379834880000} - \\ & \frac{737953655264219x^{22}}{139183101895193395200000} + \frac{32765821689796589x^{24}}{80169466691631395635200000} - \\ & \frac{3903514575224021x^{26}}{320677866766525582540800000} + \frac{49374103882823021x^{28}}{251411447544956056711987200000} - \\ & \frac{452165258793448271x^{30}}{320677866766525582540800000} + \frac{32493619195524407x^{32}}{22627030279046045104078848000000} - \\ & \frac{3338407184243827x^{34}}{320677866766525582540800000} + \frac{2425746698010469x^{36}}{2317007900574315018657674035200000} - \\ & \frac{536482233689x^{38}}{46340158011486300373153480704000000} + \frac{423385249313x^{40}}{8579549254698035040515272998912000000} + \\ & \frac{612824946764145360036805214208000000}{190656650104400778678117177753600000000} + \\ & \text{Log}[x] - \frac{1}{4}x^2 \text{Log}[x] + \frac{1}{64}x^4 \text{Log}[x] - \frac{x^6 \text{Log}[x]}{2304} + \frac{x^8 \text{Log}[x]}{147456} - \frac{x^{10} \text{Log}[x]}{14745600} + \frac{x^{12} \text{Log}[x]}{2123366400} - \\ & \frac{x^{14} \text{Log}[x]}{416179814400} + \frac{x^{16} \text{Log}[x]}{106542032486400} - \frac{x^{18} \text{Log}[x]}{34519618525593600} + \frac{x^{20} \text{Log}[x]}{13807847410237440000} \end{aligned}$$

Notice that this is definitely not a Taylor Series! Let's plot it against the known solution  $\frac{\pi}{2} Y_0(x)$  and see how they compare

```
Plot[{y2[x],  $\frac{\pi}{2}$  BesselY[v1, x]}, {x, 0, 8}, ImageSize -> {500, Automatic},
PlotStyle -> {Thick}, PlotRange -> {Automatic, {-5, 5}}, AspectRatio -> 1 / 3]
```



Not bad! Let's try this again, with more terms, and output suppressed:

```

J1tv-,k-[x-] :=  $\left(\frac{x}{2}\right)^v \sum_{m=0}^k \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma[1+v+m]}$ 

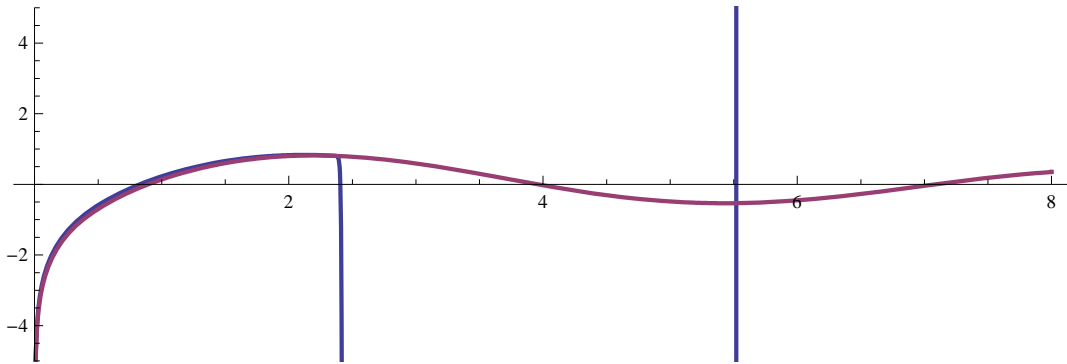
v1 = 0;
k1 = 200;

J1tv1,k1[x];

J1sq[x_] = Expand[J1tv1,k1[x]2];
inJ1sq[x_] = Normal[Series[ $\frac{1}{J1sq[x]}$ , {x, 0, 2 * k1}]];
y2[x_] =  $\left(\int inJ1sq[x] * e^{-\int \frac{1}{x} dx} dx\right) * J1t_{v1,k1}[x]$ ;

Plot[{y2[x],  $\frac{\pi}{2}$  BesselY[v1, x]}, {x, 0, 8}, ImageSize → {500, Automatic},
PlotStyle → {Thick}, PlotRange → {Automatic, {-5, 5}}, AspectRatio → 1 / 3]

```



It looks a bit better, but the radius of convergence of our power series appears to be around 2.5.