Legendre polynomials

http : // math.fullerton.edu/mathews/n2003/LegendrePolyMod.html

```
ClearAll["Global`*"];
Off[General::spell, General::spell1]
```

Legendre polynomials Pn (x) (LegendreP[n, x])satisfy the differential equation $(1-x^2)(d^2y/dx^2)-2x(dy/dx)+n(n+1)y=0$

Show the Legendre polynomials $p_0[x] = 1$,

$p_1[x] = x$, $p_2[x] = -\frac{1}{2}+\frac{3x^2}{2}$, $p_3[x] = -\frac{3x}{2}+\frac{5x^3}{2}$,

$p_4[x] = \frac{3}{8}-\frac{15x^2}{4}+\frac{35x^4}{8}$ and $p_5[x] = \frac{15x}{8}-\frac{35x^3}{4}+\frac{63x^5}{8}$

```
a = -1;
b = 1;
For[n = 0, n ≤ 5, n++,
  Pn[x_] = LegendreP[n, x];
  Print["P"n, "[x] = ", Pn[x] ]; ];
```

$P_0[x] = 1$

$P_1[x] = x$

$P_2[x] = \frac{1}{2}(-1+3x^2)$

$P_3[x] = \frac{1}{2}(-3x+5x^3)$

$P_4[x] = \frac{1}{8}(3-30x^2+35x^4)$

$P_5[x] = \frac{1}{8}(15x-70x^3+63x^5)$

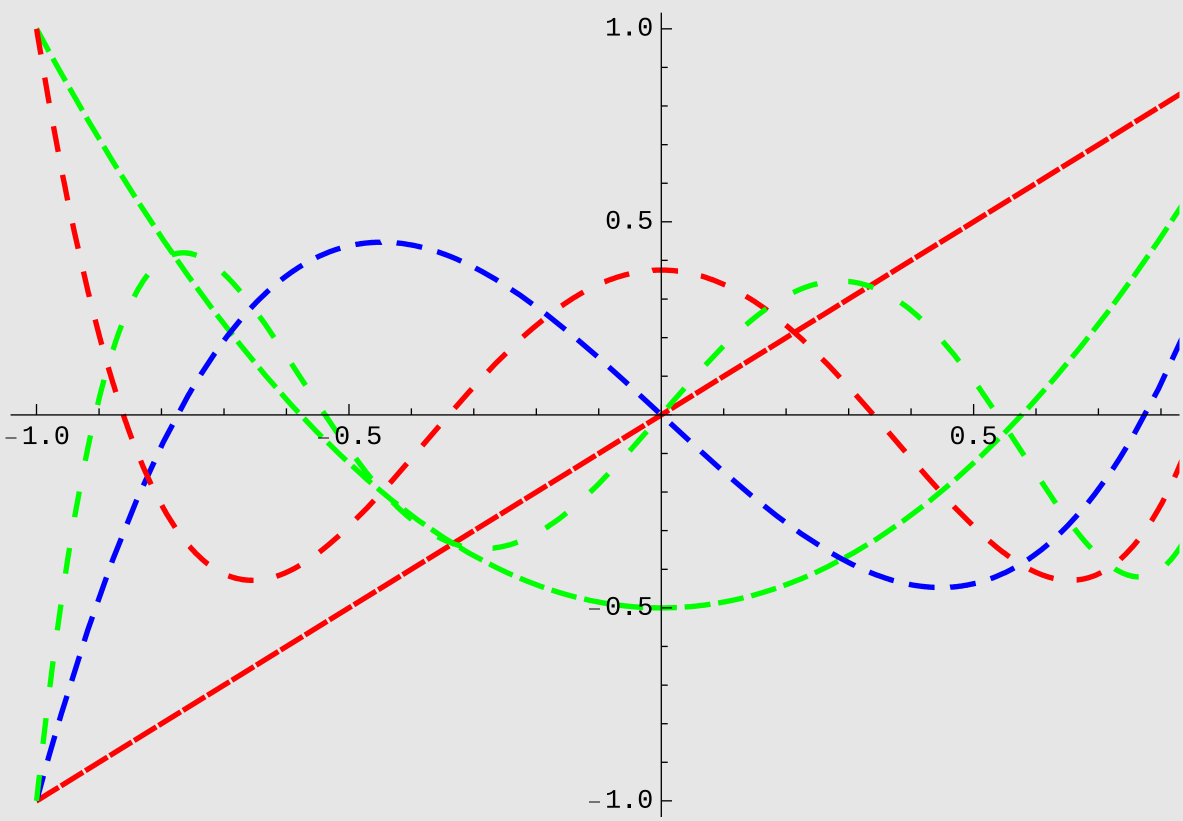
You can obtain quiqly the above list with associated Legendre polynomials

```
LegendreP[{1, 2, 3, 4,  5}, x]
```

$$\left\{x,\ \frac{1}{2}\left(-1+3x^2\right),\ \frac{1}{2}\left(-3x+5x^3\right),\ \frac{1}{8}\left(3-30x^2+35x^4\right),\ \frac{1}{8}\left(15x-70x^3+63x^5\right)\right\}$$

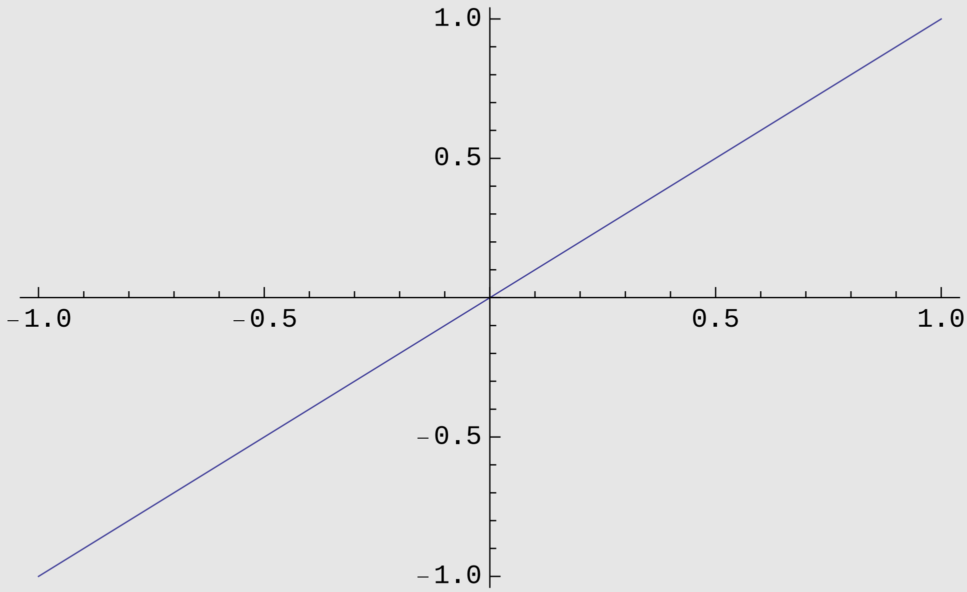
```
huelist = Table[{Hue[k / 3], Dashing[{0.02, k / 5 * 0.03}], Thick}, {k, 0, 5}];
```

```
Plot[Evaluate@Table[LegendreP[n, x], {n, 5}], {x, -1, 1},
 PlotStyle → huelist]
```



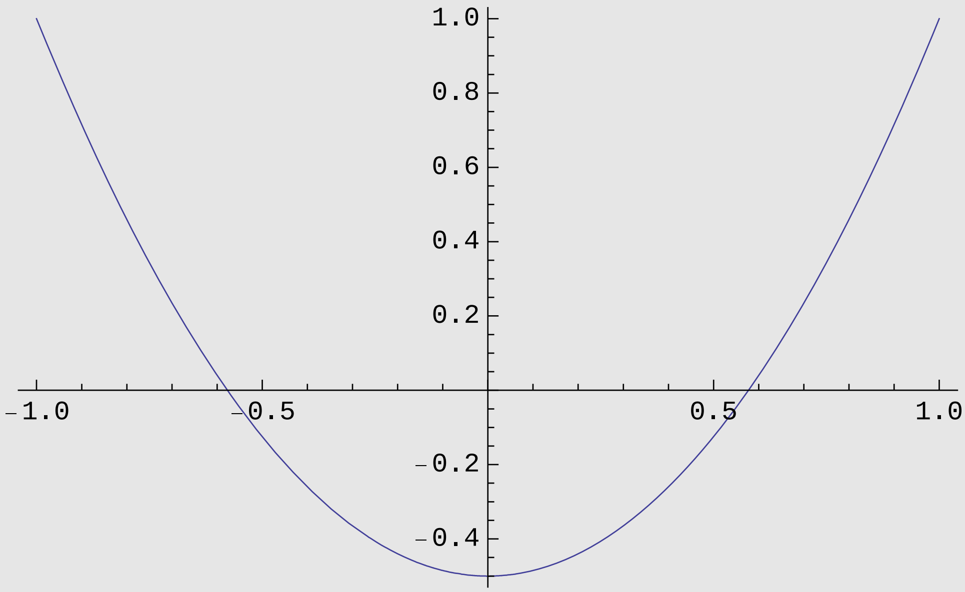
```
LegendreP[1, x]
g1 = Plot[LegendreP[1, x], {x, -1, 1}]
```

```
x
```



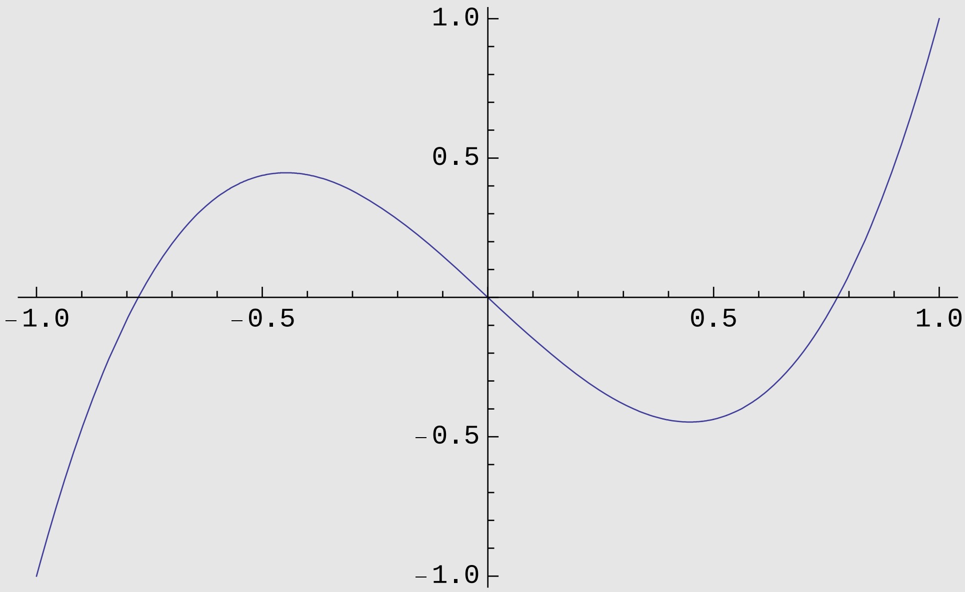
```
LegendreP[2, x]
g2 = Plot[LegendreP[2, x], {x, -1, 1}]
```

$$\frac{1}{2}\left(-1+3x^2\right)$$



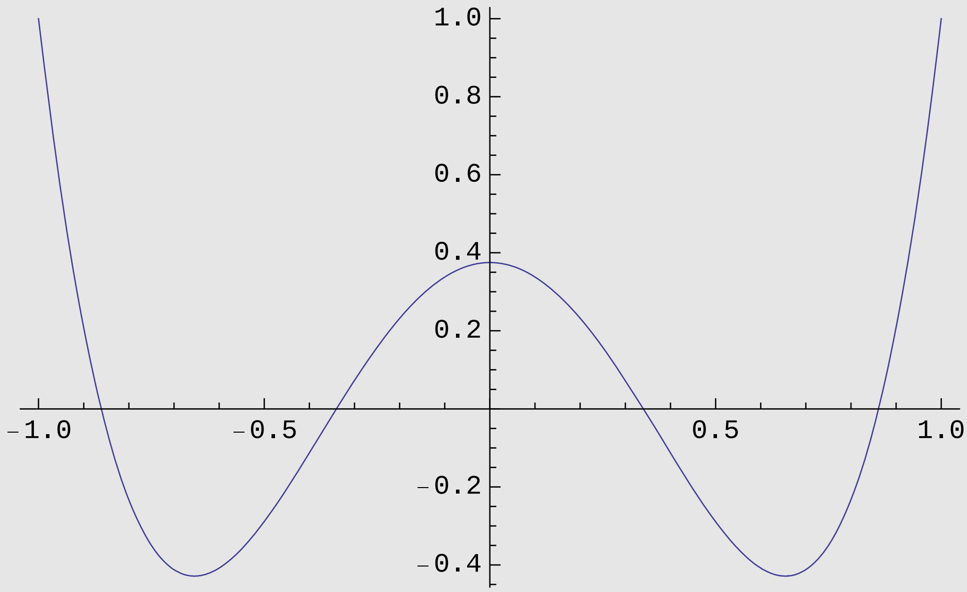
```
LegendreP[3, x]
g3 = Plot[LegendreP[3, x], {x, -1, 1}]
```

$$\frac{1}{2}\left(-3\,x+5\,x^{3}\right)$$

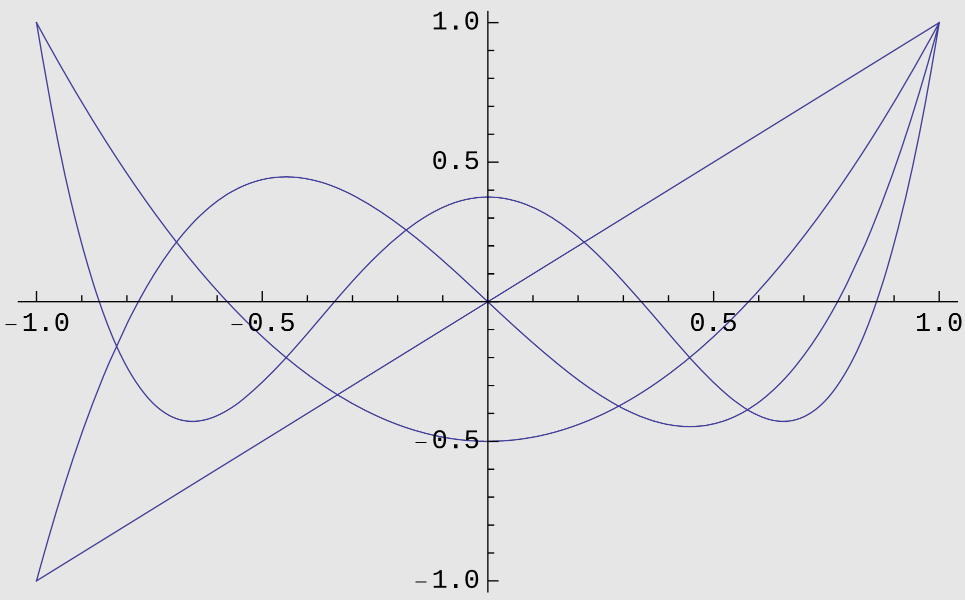


```
LegendreP[3, x]
g4 = Plot[LegendreP[4, x], {x, -1, 1}]
```

$$\frac{1}{2}\left(-3\,x+5\,x^{3}\right)$$



```
Show[{g1, g2, g3, g4}]
```



How to demonstrate the orthogonality of Legendre polynomials on [-1,1]

```
c = 5;
⟨f_, g_⟩ := ∫_a^b f[x] g[x] dx;
For[n = 0, n ≤ c - 1, n++,
  Print[""];
  For[m = n + 1, m ≤ c, m++,
   Print["⟨", "P"_n, "[x],", "P"_m, "[x]⟩ = ", "∫_a^b", "(", P_n[x], ")(",
     P_m[x], ")dx = ", ⟨P_n, P_m⟩]; ]; ];
```

$\langle P_0[x], P_1[x]\rangle = \int_{-1}^{1} (1)(x)\,dx = 0$

$\langle P_0[x], P_2[x]\rangle = \int_{-1}^{1} (1)\left(\frac{1}{2}\left(-1+3x^2\right)\right)dx = 0$

$\langle P_0[x], P_3[x]\rangle = \int_{-1}^{1} (1)\left(\frac{1}{2}\left(-3x+5x^3\right)\right)dx = 0$

$$\langle P_0[x], P_4[x]\rangle = \int_{-1}^{1} (1) \left(\frac{1}{8}\left(3 - 30 x^2 + 35 x^4\right)\right) dx = 0$$

$$\langle P_0[x], P_5[x]\rangle = \int_{-1}^{1} (1) \left(\frac{1}{8}\left(15 x - 70 x^3 + 63 x^5\right)\right) dx = 0$$

$$\langle P_1[x], P_2[x]\rangle = \int_{-1}^{1} (x) \left(\frac{1}{2}\left(-1 + 3 x^2\right)\right) dx = 0$$

$$\langle P_1[x], P_3[x]\rangle = \int_{-1}^{1} (x) \left(\frac{1}{2}\left(-3 x + 5 x^3\right)\right) dx = 0$$

$$\langle P_1[x], P_4[x]\rangle = \int_{-1}^{1} (x) \left(\frac{1}{8}\left(3 - 30 x^2 + 35 x^4\right)\right) dx = 0$$

$$\langle P_1[x], P_5[x]\rangle = \int_{-1}^{1} (x) \left(\frac{1}{8}\left(15 x - 70 x^3 + 63 x^5\right)\right) dx = 0$$

$$\langle P_2[x], P_3[x]\rangle = \int_{-1}^{1} \left(\frac{1}{2}\left(-1 + 3 x^2\right)\right) \left(\frac{1}{2}\left(-3 x + 5 x^3\right)\right) dx = 0$$

$$\langle P_2[x], P_4[x]\rangle = \int_{-1}^{1} \left(\frac{1}{2}\left(-1 + 3 x^2\right)\right) \left(\frac{1}{8}\left(3 - 30 x^2 + 35 x^4\right)\right) dx = 0$$

$$\langle P_2[x], P_5[x]\rangle = \int_{-1}^{1} \left(\frac{1}{2}\left(-1 + 3 x^2\right)\right) \left(\frac{1}{8}\left(15 x - 70 x^3 + 63 x^5\right)\right) dx = 0$$

$$\langle P_3[x], P_4[x]\rangle = \int_{-1}^{1} \left(\frac{1}{2}\left(-3 x + 5 x^3\right)\right) \left(\frac{1}{8}\left(3 - 30 x^2 + 35 x^4\right)\right) dx = 0$$

$$\langle P_3[x], P_5[x]\rangle = \int_{-1}^{1} \left(\frac{1}{2}\left(-3 x + 5 x^3\right)\right) \left(\frac{1}{8}\left(15 x - 70 x^3 + 63 x^5\right)\right) dx = 0$$

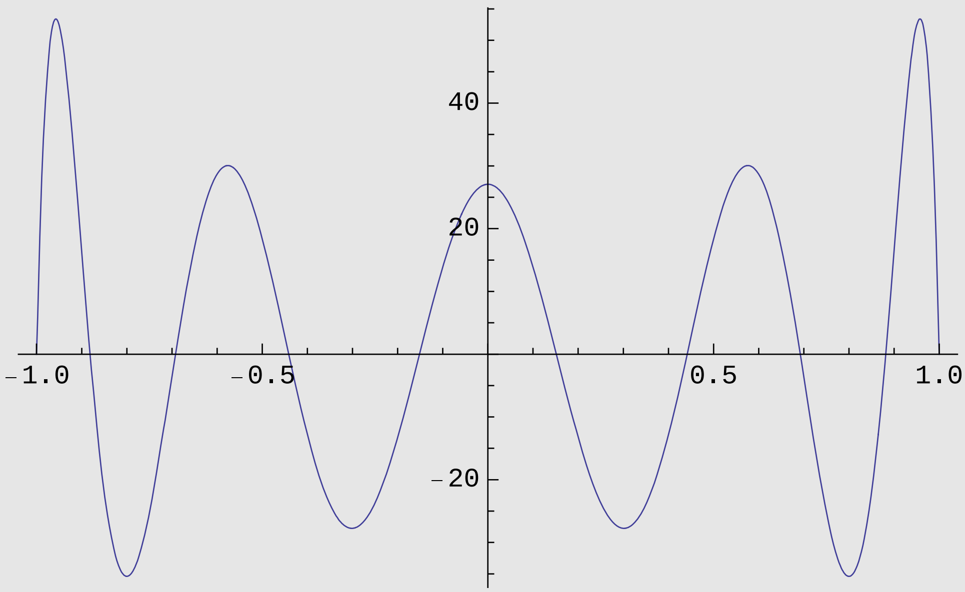
$$\langle P_4[x], P_5[x]\rangle = \int_{-1}^{1} \left(\frac{1}{8}\left(3 - 30 x^2 + 35 x^4\right)\right) \left(\frac{1}{8}\left(15 x - 70 x^3 + 63 x^5\right)\right) dx = 0$$

Associated Legendre polynomials (LegendreP[n, m, x]) are defined by $P_n^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \left(d^m / d x^m\right) P_n(x)$

```
LegendreP[2, 2, x]
```

$-3\left(-1+x^{2}\right)$

```
Plot[LegendreP[10, 2, x], {x, -1, 1}]
```



For arbitrary complex values of n, m and z, **LegendreP[n,z]** and **LegendreP[n,m,z]** give Legendre functions of the first kind.

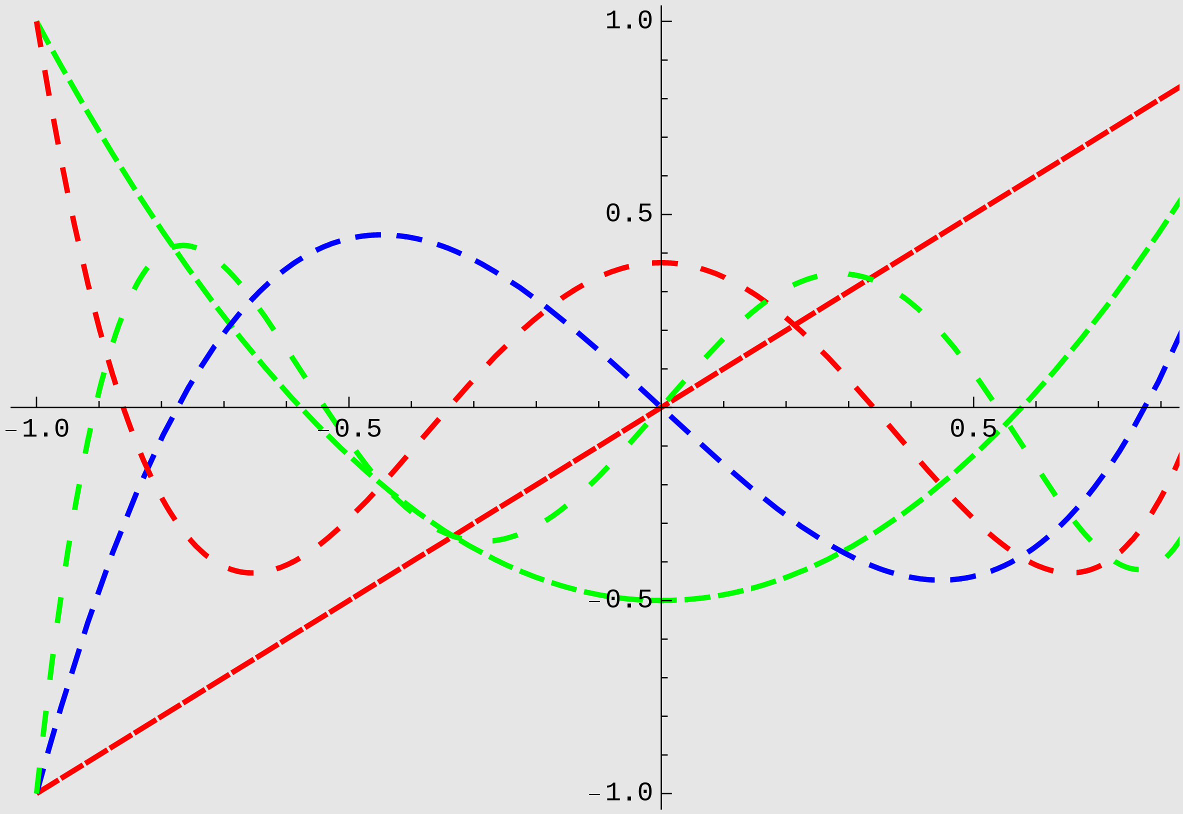
If you need to evaluate for complex orders and arguments:

```
LegendreP[5 / 2 + I, 1.5 - I]
```

13.1073 - 7.61039 ⅈ

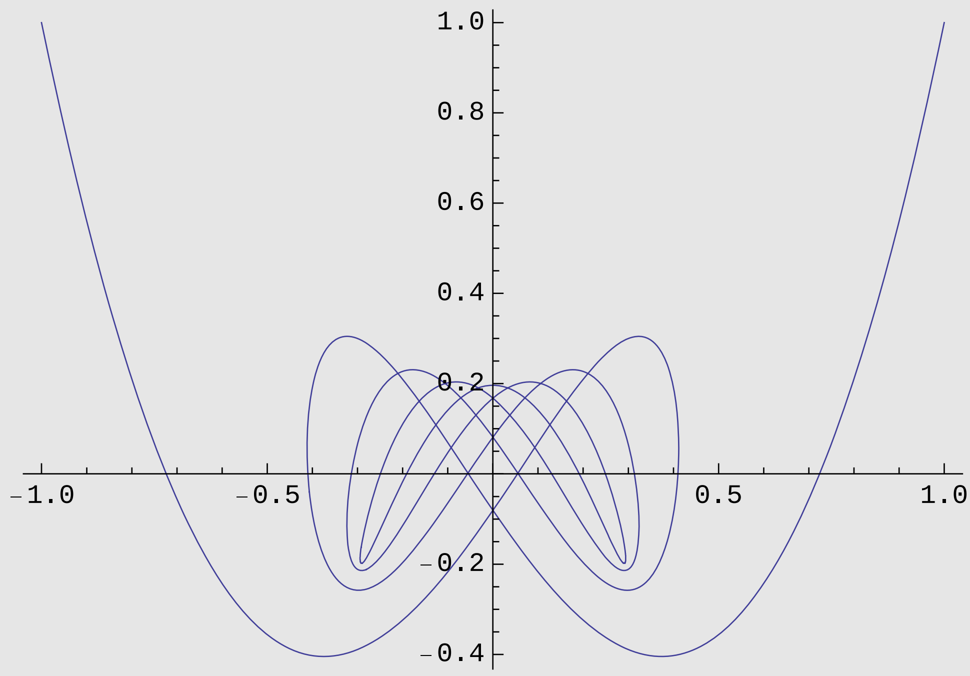
```
huelist = Table[{Hue[k / 3], Dashing[{0.02, k / 5 * 0.03}], Thick}, {k, 0, 5}];
```

```
Plot[Evaluate@Table[LegendreP[n, x], {n, 5}], {x, -1, 1},
 PlotStyle → huelist]
```



Using LegendreP[n, z] we can obtain the eneralized Lissajous figures :

```
ParametricPlot[ {LegendreP[7, x], LegendreP[16, x]}, {x, -1, 1}]
```



If you want to obtain the traditional form of associated Legendre polynomials, you can use **(//TraditionalForm)**

```
LegendreP[n, m, z] // TraditionalForm
```

0

LegendreP can be applied to a power series :

```
S1 = Series[Sin[x] * Cos[X], {x, 0, 3}]
```

$$\text{Cos}[X]\, x - \frac{1}{6}\text{Cos}[X]\, x^3 + O[x]^4$$

```
LegendreP[3 / 2, 1 / 2, S1]
```

$$-\sqrt{\frac{2}{\pi}} + \frac{7\,\text{Cos}[X]^2\,x^2}{2\sqrt{2\,\pi}} + O[x]^4$$

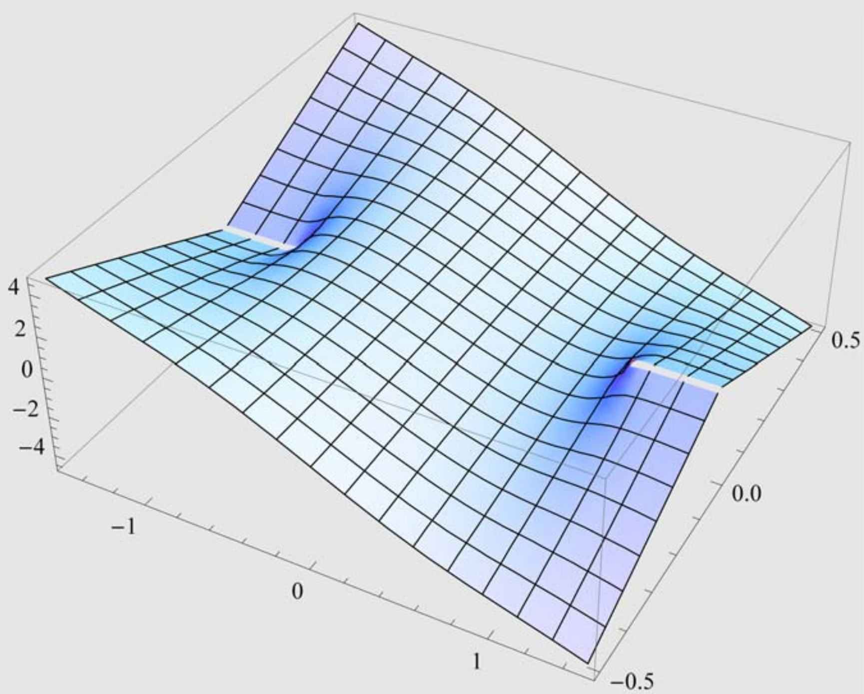
Different **LegendreP** types give different symbolic forms:

```
LegendreP[2, 1, {1, 2, 3}, z]
```

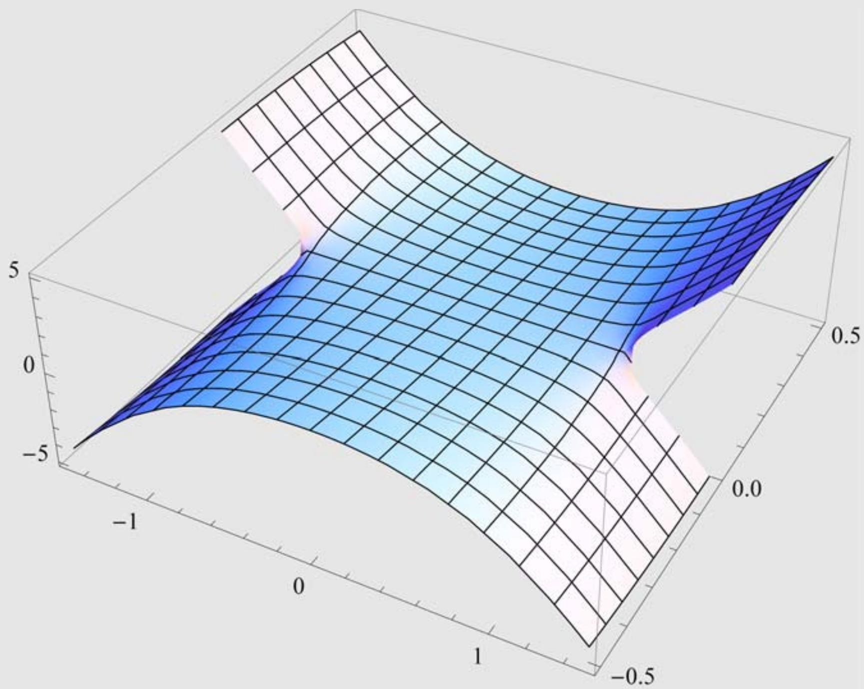
$$\left\{-3\,z\,\sqrt{1-z^2},\ -3\,\sqrt{1-z}\,z\,\sqrt{1+z},\ -\frac{3\,(1-z)\,z\,\sqrt{1+z}}{\sqrt{-1+z}}\right\}$$

having different branch cut structures :

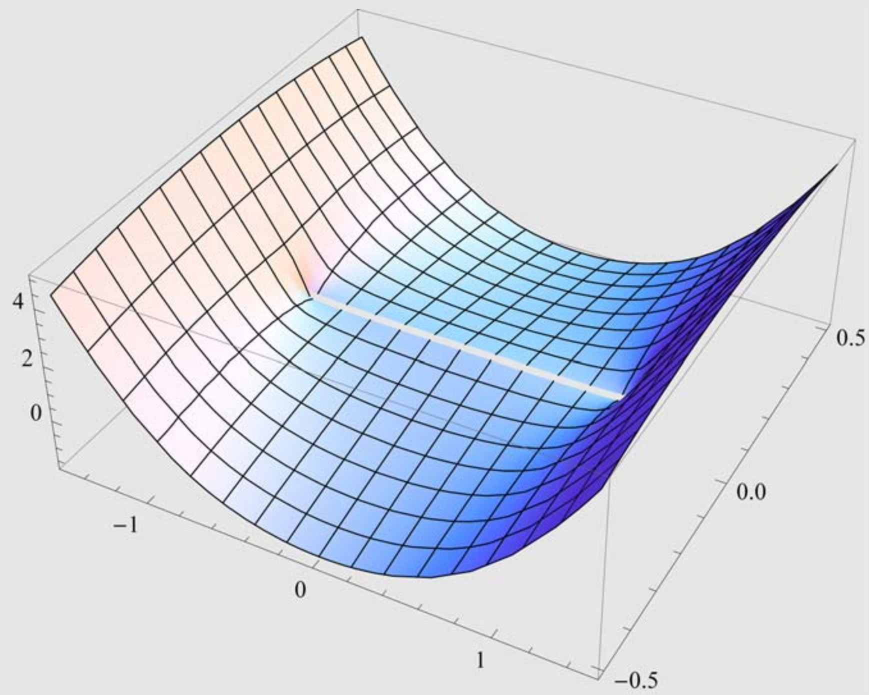
```
Plot3D[Re[LegendreP[2, 1, 2, x + I y]], {x, -1.5, 1.5}, {y, -0.5, 0.5},
 Exclusions → {{y == 0, Abs[x] > 1}}]
```



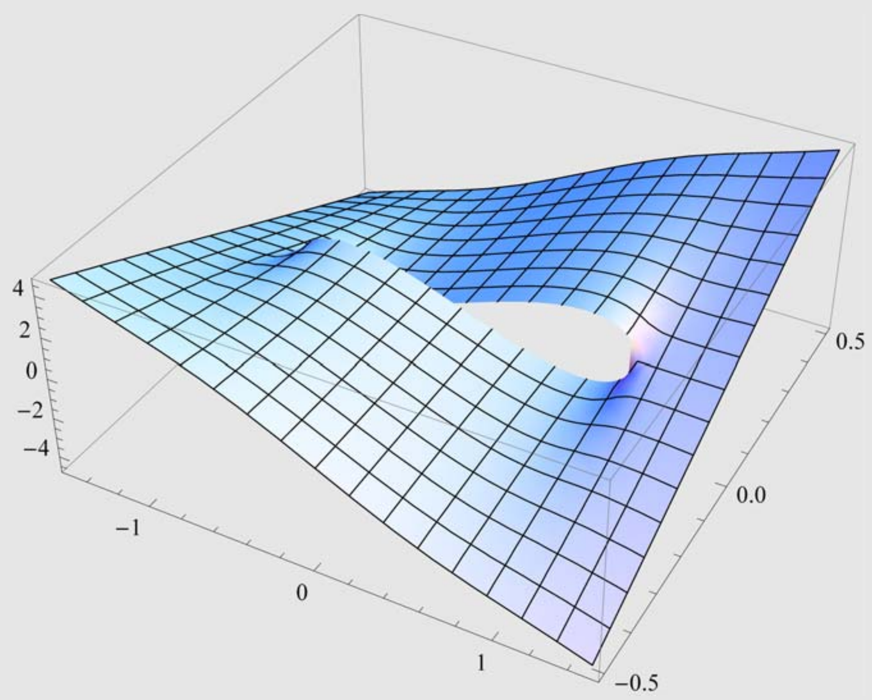
```
Plot3D[Im[LegendreP[2, 1, 2, x + I y]], {x, -1.5, 1.5}, {y, -0.5, 0.5},
 Exclusions → {{y == 0, Abs[x] > 1}}]
```



```
Plot3D[Re[LegendreP[2, 1, 3, x + I y]], {x, -1.5, 1.5}, {y, -0.5, 0.5},
 Exclusions → {{y == 0, -1 < x < 1}}]
```



```
Plot3D[Im[LegendreP[2, 1, 3, x + I y]], {x, -1.5, 1.5}, {y, -0.5, 0.5},
 Exclusions → {{y == 0, -1 < x < 1}}]
```



LegendreP[n,z] can do generalized Fourier transform for functions on interval - 1 to 1 :

```
Table[(n + 1 / 2) Integrate[LegendreP[n, x] Sin[Pi x], {x, -1, 1}],
 {n, 0, 4}]
```

$$\left\{0,\ \frac{3}{\pi},\ 0,\ \frac{7\left(-15+\pi^2\right)}{\pi^3},\ 0\right\}$$

```
Plot[{Sin[Pi x], %.Table[LegendreP[n - 1, x], {n, Length[%]}]},
 {x, -1, 1}]
```