

Frobenius Series

```
ClearAll["Global`*"];  
Off[General::spell, General::spell1]
```

Use Frobenius series to solve the D. E.

$$x^2 y''[x] + xy'[x] + (x^2 - 1)y[x] = 0$$

Determine the nature of the singularity at x_0

```

X0 = 0;
P[x_] = x^2;
Q[x_] = x;
R[x_] = (x^2 - 1);
f1[x_] =  $\frac{Q[x]}{P[x]}$ ;
f2[x_] =  $\frac{R[x]}{P[x]}$ ;
deq = P[x] * y''[x] + Q[x] * y'[x] + R[x] * y[x] == 0;
Print[deq];
If[P[X0] == 0,
  Print["P[x]=", P[x]];
  Print["P[" , X0, "]=", P[X0]];
  Print["The point x0= ", X0, " is a singular point."];
  Print["Proceed with a Frobenius solution"];
  Print["The point x0=", X0, "is not a singular point."];
  Print["Use the power series method for an arbitrary point."];
  Print["y''[x]+f1[x]y'[x]+f2[x]y[x]==0"];
  Print["f1[x]= $\frac{Q[x]}{P[x]}$ =", Together[ $\frac{Q[x]}{P[x]}$ ], "=",
Series[f1[x], {x, X0, 3}]];
  Print["f2[x]= $\frac{R[x]}{P[x]}$ =", Together[ $\frac{R[x]}{P[x]}$ ], "=",
Series[f2[x], {x, X0, 3}]];
  If[X0 > 0, Print["y[x]= $\sum_{n=0}^{\infty} a_n x^n$ ", ", etc."]];
  If[X0 == 0, Print["y[x]= $\sum_{n=0}^{\infty} a_n x^n$ ", "(x-", X0, ")^n ", ", etc."]];
  If[X0 < 0, Print["y[x]= $\sum_{n=0}^{\infty} a_n x^n$ ", "(x+", X0, ")^n ", ", etc."]];];

```

$$(-1 + x^2) y[x] + x y'[x] + x^2 y''[x] == 0$$

$$P[x]=x^2$$

$$P[0]=0$$

The point $x_0= 0$ is a singular point.

Proceed with a Frobenius solution

The point $x_0=0$ is not a singular point.

Use the power series method for an arbitrary point.

$$y''[x] + f_1[x]y'[x] + f_2[x]y[x] = 0$$

$$f_1[x] = \frac{Q[x]}{P[x]} = -\frac{1}{x} - \frac{1}{x^2} + 1 + O[x]^4$$

$$f_2[x] = \frac{R[x]}{P[x]} = \frac{-1 + x^2}{x^2} = f_2[0] + (f_2)'[0]x + \frac{1}{2}(f_2)''[0]x^2 + \frac{1}{6}(f_2)^{(3)}[0]x^3 + O[x]^4$$

$$y[x] = \sum_{n=0}^{\infty} a_n x^n (x-0)^n, \text{ etc.}$$

Now we must construct the indicial equation

```

X0 = 0;
phi1[x_] = x *  $\frac{Q[x]}{P[x]}$ ;
phi2[x_] = x^2 *  $\frac{R[x]}{P[x]}$ ;
p0 = Limit[x *  $\frac{Q[x]}{P[x]}$ , x -> X0];
q0 = Limit[x^2 *  $\frac{R[x]}{P[x]}$ , x -> X0];
Clear[r];
indeq = r * (r - 1) + p0 * r + q0 == 0;
Print["P[x]=", P[x]];
Print["Q[x]=", Q[x]];
Print["R[x]=", R[x], "\n"];
Print["phi1[x_] = x *  $\frac{Q[x]}{P[x]}$  = ", x *  $\frac{Q[x]}{P[x]}$ ];
Print["p0=", "lim", " x *  $\frac{Q[x]}{P[x]}$  = ", p0];
Print[" "];
Print["phi2[x_] = x^2 *  $\frac{R[x]}{P[x]}$  = ", x^2 *  $\frac{R[x]}{P[x]}$ ];
Print["q0=", "lim", " x^2 *  $\frac{R[x]}{P[x]}$  = ", q0];
Print["\n", "The indicial equation is"];
Print["\n", "r(r-1)+p0r+q0= 0"];
Print[indeq];
indeq = MapAll[Expand, indeq];
Print[indeq];

```

$$P[x] = x^2$$

$$Q[x] = x$$

$$R[x] = -1 + x^2$$

$$\phi_1[x_] = x \frac{Q[x]}{P[x]} = 1$$

$$p_0 = \lim_{x \rightarrow X_0} x \frac{Q[x]}{P[x]} = 1$$

$$\phi_2[x_] = x^2 \frac{R[x]}{P[x]} = -1 + x^2$$

$$q_0 = \lim_{x \rightarrow x_0} x^2 * \frac{R[x]}{P[x]} = -1$$

The indicial equation is

$$r(r-1) + p_0 r + q_0 = 0$$

$$-1 + r + (-1 + r) r = 0$$

$$-1 + r^2 = 0$$

Now we must find the roots of the indicial equation

```
Print["The indicial equation is"];
Print[indeq];
Print["The roots of the indicial equation are"];
solind = Solve[indeq, r];
r1 = solind[[2,1,2]];
r2 = solind[[1,1,2]];
Print["r1 =", r1];
Print["r2 =", r2];
```

The indicial equation is

$$-1 + r^2 = 0$$

The roots of the indicial equation are

$$r_1 = 1$$

$$r_2 = -1$$

Now we form the first Frobenius solution corresponding to the root r_1

```

Clear[c, k, n, s, x];
c = 0; Remove[c];
r = 1;
n = 10;

s[x_] = Expand[x^r * (Sum[c_k * x^k, {k, 0, n}]) + O[x]^{n+1}];

coeff = Table[c_k, {k, 1, n}];
Print[" s[x] = ", s[x]];
Print[" s'[x] = ", s'[x]];
Print[" s''[x] = ", s''[x]];
Print[" If the derivative terms are substituted into"];
Print["x^2 s''[x] + x s'[x] + (x^2 - 1) s[x] = 0"];
Print[" We get"];
deq = Expand[x^2 * s''[x] + x * s'[x] + (x^2 - 1) * s[x]] == 0;
Print[deq];

```

$$s[x] = c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + c_5 x^6 + c_6 x^7 + c_7 x^8 + c_8 x^9 + c_9 x^{10} + O[x]^{11}$$

$$s'[x] = c_0 + 2 c_1 x + 3 c_2 x^2 + 4 c_3 x^3 + 5 c_4 x^4 + 6 c_5 x^5 + 7 c_6 x^6 + 8 c_7 x^7 + 9 c_8 x^8 + 10 c_9 x^9 + O[x]^{10}$$

$$s''[x] = 2 c_1 + 6 c_2 x + 12 c_3 x^2 + 20 c_4 x^3 + 30 c_5 x^4 + 42 c_6 x^5 + 56 c_7 x^6 + 72 c_8 x^7 + 90 c_9 x^8 + O[x]^9$$

If the derivative terms are substituted into

$$x^2 s''[x] + x s'[x] + (x^2 - 1) s[x] = 0$$

We get

$$3 c_1 x^2 + (c_0 + 8 c_2) x^3 + (c_1 + 15 c_3) x^4 + (c_2 + 24 c_4) x^5 + (c_3 + 35 c_5) x^6 + (c_4 + 48 c_6) x^7 + (c_5 + 63 c_7) x^8 + (c_6 + 80 c_8) x^9 + (c_7 + 99 c_9) x^{10} + O[x]^{11} = 0$$

Now we form the set of equations to solve and do it

```

eqns = LogicalExpand[deq];
Print[deq];
seteq = Print[TableForm[ReplaceAll[eqns, And -> List]]];
solset = Solve[eqns, coeff];
Print[TableForm[Sort[solset[[1]]]];
s[x_] = ReplaceAll[s[x], solset[[1]]];
Print["y =", s[x]];
Print["y =", Normal[s[x]], " + ..."];

```

$$3 c_1 x^2 + (c_0 + 8 c_2) x^3 + (c_1 + 15 c_3) x^4 + (c_2 + 24 c_4) x^5 + (c_3 + 35 c_5) x^6 + (c_4 + 48 c_6) x^7 + (c_5 + 63 c_7) x^8 + (c_6 + 80 c_8) x^9 + (c_7 + 99 c_9) x^{10} + O[x]^{11} = 0$$

$$\begin{aligned}
3 c_1 &= 0 \\
c_0 + 8 c_2 &= 0 \\
c_1 + 15 c_3 &= 0 \\
c_2 + 24 c_4 &= 0 \\
c_3 + 35 c_5 &= 0 \\
c_4 + 48 c_6 &= 0 \\
c_5 + 63 c_7 &= 0 \\
c_6 + 80 c_8 &= 0 \\
c_7 + 99 c_9 &= 0
\end{aligned}$$

Solve::svars: Equations may not give solutions for all "solve" variables. >>

$$\begin{aligned}
c_1 &\rightarrow 0 \\
c_2 &\rightarrow -\frac{c_0}{8} \\
c_3 &\rightarrow 0 \\
c_4 &\rightarrow \frac{c_0}{192} \\
c_5 &\rightarrow 0 \\
c_6 &\rightarrow -\frac{c_0}{9216} \\
c_7 &\rightarrow 0 \\
c_8 &\rightarrow \frac{c_0}{737280} \\
c_9 &\rightarrow 0
\end{aligned}$$

$$y = c_0 x - \frac{c_0 x^3}{8} + \frac{c_0 x^5}{192} - \frac{c_0 x^7}{9216} + \frac{c_0 x^9}{737280} + O[x]^{11}$$

$$y = x c_0 - \frac{x^3 c_0}{8} + \frac{x^5 c_0}{192} - \frac{x^7 c_0}{9216} + \frac{x^9 c_0}{737280} + \dots$$

The first Frobenius solution is:

```

s1[x_] = ReplaceAll[s[x], c0 -> 1];
Print["f1[x] =", s1[x]];
Print["f1[x] =", Normal[s1[x]], " +..."];
Print["y =", (x^r), "(", Normal[x^-r * s1[x]], " +...)"];

```

$$f1[x] = x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{9216} + \frac{x^9}{737280} + O[x]^{11}$$

$$f1[x] = x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{9216} + \frac{x^9}{737280} + \dots$$

$$y = x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} + \dots \right)$$

This first Frobenius series solution is a Bessel Function of the first kind ($J[1,x]$) defined at the origin x_0

```

x = 0; y = 0; Remove[x, y];

```



```

deq = x2 * y''[x] + x * y'[x] + (x2 - 1) * y[x] == 0;
sol1 = DSolve[deq, y[x], x];
Print[deq];
Print[sol1];
Print[" The two linearly independent solutions are", "\n"];
b1[x_] = BesselJ[1, x];
Print["b1[x] =", b1[x]];
b2[x_] = BesselY[1, x];
Print[" "];
Print["b2[x] =", b2[x]];

```

```

(-1 + x2) y[x] + x y'[x] + x2 y''[x] == 0
{{y[x] -> BesselJ[1, x] C[1] + BesselY[1, x] C[2]}}
The two linearly independent solutions are

b1[x] =BesselJ[1, x]

b2[x] =BesselY[1, x]

```

The second solution to the differential equation is a Bessel Function of the Second Kind (Y[1,x]). It is not a simple Frobenius series and will be discussed in another module. At this time we could plot the series approximations and the analytic solutions. To see the difference in the graphs we will reduce the number of terms in the series.

```

S1[x_] = Take[Normal[s1[x]], 3]
Needs["Graphics`Colors`"]
Plot[{S1[x], 2 * b1[x]}, {x, 0, 5}, PlotStyle -> {Blue, Red}]
Print["S1[x] =", S1[x]];
Print["b1[x] =", b1[x]];

```

$$x - \frac{x^3}{8} + \frac{x^5}{192}$$

General::obspkg :

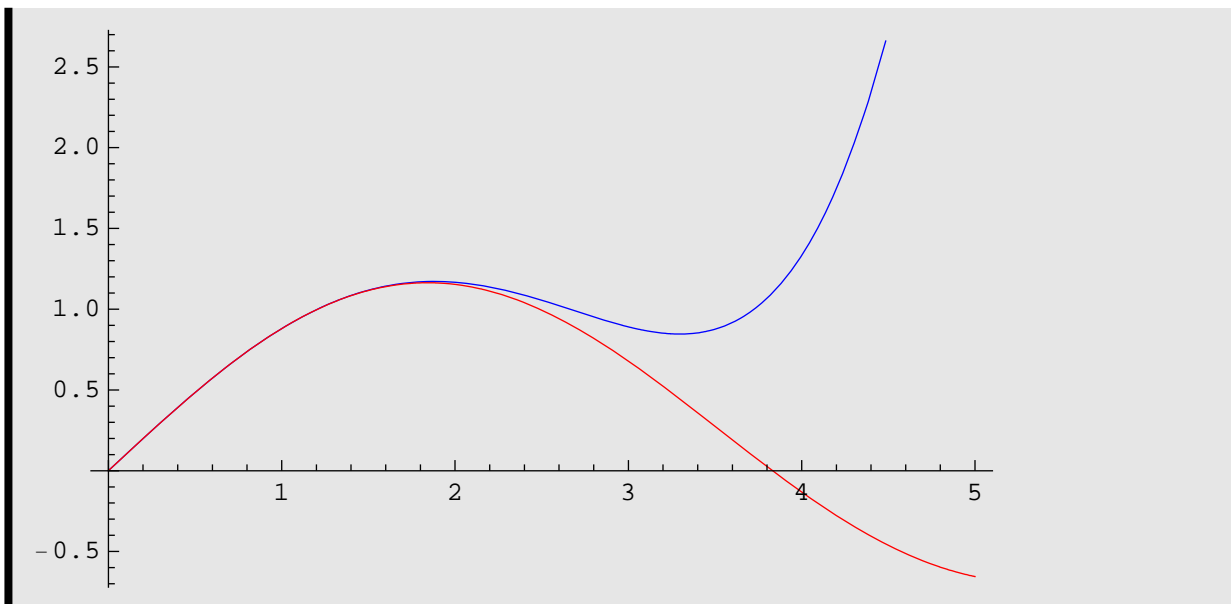
Graphics`Colors` is now obsolete. The legacy version being loaded may conflict with current *Mathematica* functionality. See the Compatibility Guide for updating information. >>

LightBlue::shdw : Symbol LightBlue appears in multiple contexts {Graphics`Colors`, System`}; definitions in context Graphics`Colors` may shadow or be shadowed by other definitions. >>

LightGray::shdw : Symbol LightGray appears in multiple contexts {Graphics`Colors`, System`}; definitions in context Graphics`Colors` may shadow or be shadowed by other definitions. >>

LightPink::shdw : Symbol LightPink appears in multiple contexts {Graphics`Colors`, System`}; definitions in context Graphics`Colors` may shadow or be shadowed by other definitions. >>

LightYellow::shdw : Symbol LightYellow appears in multiple contexts {Graphics`Colors`, System`}; definitions in context Graphics`Colors` may shadow or be shadowed by other definitions. >>



$$S1[x] = x - \frac{x^3}{8} + \frac{x^5}{192}$$

```
b1[x] =BesselJ[1, x]
```

We try to find the recursive formula for the coefficients

If we look at the series in more depth we will be able to obtain the analytic solutions as infinite sums. First find the recursive formula for the coefficients of x^{r+n} . If you try this be sure to use the " := " replacement delayed structure to avoid an infinite recursion.

Also, include the semicolon at the end of the lines.

```

Clear[c, k, n, r, t, x];
c = 0; Remove[c];
t[x_] = Sum[c_k * x^{r+k}, {k, n-2, n};
termN = x^2 * t'[x] + x * t'[x] + (x^2 - 1) * t[x] == 0;
termN = ExpandAll[termN];
Print["The general term in the sum will be:", "\n"];
Print[termN];
Print["\n", "Now we must simplify it!", "\n"];
termN = termN /. x^{r+n} -> 1 /. x^{-1+r+n} -> 0 /. x^{-2+r+n} -> 0 /. x^{1+r+n} -> 0 /. x^{2+r+n} -> 0;
Print[termN];
Print["Solve it"];
solset = Solve[termN, c_n];
Print["\n", "Solve it for the recursive formula", "\n"];
Print[solset[[1,1]]];

```

The general term in the sum will be:

$$\begin{aligned}
& 3x^{-2+n+r} c_{-2+n} - 4nr x^{-2+n+r} c_{-2+n} + n^2 x^{-2+n+r} c_{-2+n} - 4r x^{-2+n+r} c_{-2+n} + \\
& 2nr x^{-2+n+r} c_{-2+n} + r^2 x^{-2+n+r} c_{-2+n} + x^{n+r} c_{-2+n} - 2nr x^{-1+n+r} c_{-1+n} + \\
& n^2 x^{-1+n+r} c_{-1+n} - 2r x^{-1+n+r} c_{-1+n} + 2nr x^{-1+n+r} c_{-1+n} + r^2 x^{-1+n+r} c_{-1+n} + \\
& x^{1+n+r} c_{-1+n} - x^{n+r} c_n + n^2 x^{n+r} c_n + 2nr x^{n+r} c_n + r^2 x^{n+r} c_n + x^{2+n+r} c_n = 0
\end{aligned}$$

Now we must simplify it!

$$c_{-2+n} - c_n + n^2 c_n + 2nr c_n + r^2 c_n = 0$$

Solve it

Solve it for the recursive formula

$$c_n \rightarrow -\frac{c_{-2+n}}{-1+n^2+2nr+r^2}$$

Now look at each series individually. The first Frobenius series corresponding to $r_1 = 1$ is:

```

Clear[c, k, n, x];
c = 0; Remove[c];
r = 1;
c0 = 1;
c1 = 0;
cn :=  $\frac{-c_{n-2}}{-1+n^2+2*n*r+r^2}$ ;
f1[x_] = Expand[xr *  $\left(\sum_{k=0}^{13} c_k * x^k\right)$ ];
For[i = 0, i ≤ 10, i++, Print["c"i, " = ", ci]];
Print["cn", Simplify[ $\frac{-c_{n-2}}{-1+n^2+2*n*r+r^2}$ ]];
Print["f1[x] =", f1[x]];

```

$$c_0 = 1$$

$$c_1 = 0$$

$$c_2 = -\frac{1}{8}$$

$$c_3 = 0$$

$$c_4 = \frac{1}{192}$$

$$c_5 = 0$$

$$c_6 = -\frac{1}{9216}$$

$$c_7 = 0$$

$$c_8 = \frac{1}{737280}$$

$$c_9 = 0$$

$$c_{10} = -\frac{1}{88473600}$$

$$c_n = \frac{c_{n-2}}{2n+n^2}$$

$$f1[x] = x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{9216} + \frac{x^9}{737280} - \frac{x^{11}}{88473600} + \frac{x^{13}}{14863564800}$$

The coefficient can be expressed in closed for-

m:

$$C_n = \frac{(-1)^{n-1}}{2^{t n-t} (n!) (n-1)!}$$

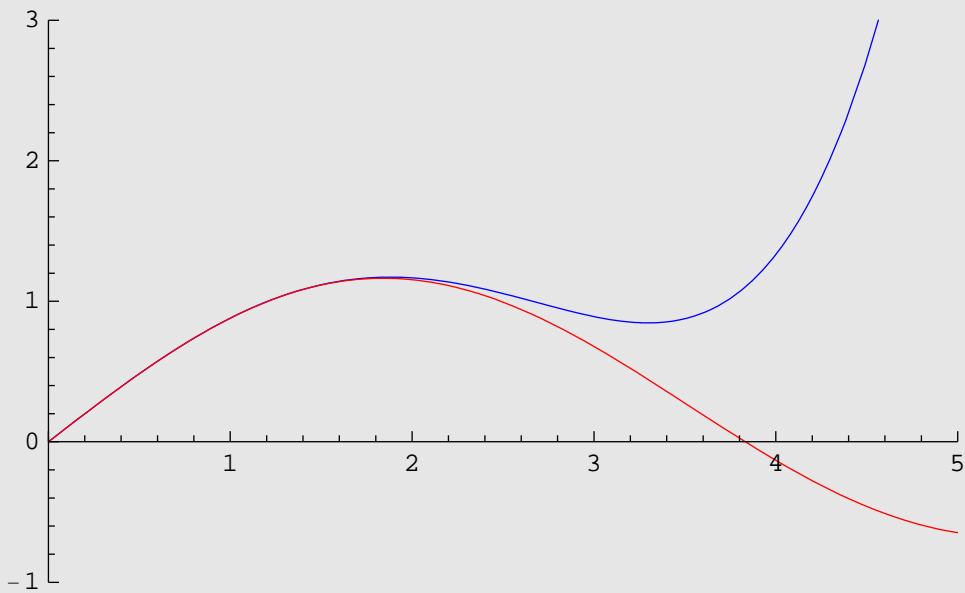
```
Table[ $\frac{(-1)^{n-1}}{2^{2*n-2} * (n!) * (n-1)!}$ , {n, 1, 7}]
```

```
{1, - $\frac{1}{8}$ ,  $\frac{1}{192}$ , - $\frac{1}{9216}$ ,  $\frac{1}{737280}$ , - $\frac{1}{88473600}$ ,  $\frac{1}{14863564800}$ }
```

```

S1[x_] = Take[Normal[s1[x]], 3];
Needs["Graphics`Colors`"];
Plot[{S1[x], f1[x]}, {x, 0, 5}, PlotRange -> {{0, 5}, {-1, 3}},
  PlotStyle -> {Blue, Red}]
Print["S1[x] =", S1[x]];
Print["f1[x] =", f1[x]];

```



$$S1[x] = x - \frac{x^3}{8} + \frac{x^5}{192}$$

$$f1[x] = x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{9216} + \frac{x^9}{737280} - \frac{x^{11}}{88473600} + \frac{x^{13}}{14863564800}$$

The the first Frobenius series solution is twice the Bessel function `BesselJ[1,x]`

```

J1[x_] = Sum[(-1)^(n-1) * x^(2*n-1) / (2^(2*n-1) * (n!) * (n-1)!), {n, 1, Infinity};

Print["J1[x] = Sum[(-1)^(n-1) * x^(2*n-1) / (2^(2*n-1) * (n!) * (n-1)!), {n, 1, 7} + ..."];

Print["J1[x] = ", Sum[(-1)^(n-1) * x^(2*n-1) / (2^(2*n-1) * (n!) * (n-1)!), {n, 1, 7}, "+ ..."];

Print["J1[x] = Sum[(-1)^(n-1) * x^(2*n-1) / (2^(2*n-1) * (n!) * (n-1)!), {n, 1, Infinity}"];

Print["J1[x] = ", Sum[(-1)^(n-1) * x^(2*n-1) / (2^(2*n-1) * (n!) * (n-1)!), {n, 1, Infinity}];

```

$$J_1[x] = \sum_{n=1}^7 \frac{(-1)^{n-1}}{2^{2*n-1} * (n!) * (n-1)!} x^{2*n-1} + \dots$$

$$J_1[x] = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \frac{x^9}{1474560} - \frac{x^{11}}{176947200} + \frac{x^{13}}{29727129600} + \dots$$

$$J_1[x] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2*n-1} * (n!) * (n-1)!} x^{2*n-1}$$

$$J_1[x] = \text{BesselJ}[1, x]$$

The Frobenius series solution $s(x)$ will have the initial conditions $s(0) = 0$ will, $s'(0) = 1/2$, whereas the the built in Mathematica function `BesselJ[1, x]` does not have the initial value 0 because it is the negative of the derivative of `BesselJ[0, x]`.


```
f[x_] = BesselJ[1, x];
Print["∂xBesselJ[0,x]=", ∂xBesselJ[0, x], "\n"]
Print["f[x]=", f[x]];
Print["f[0]=", f[0]];
Print["and"];
Print["f'[x]=∂xBesselJ[1,x]="];
Print["f'[x]=", f'[x]];
Print["f'[0]=", f'[0]];
```

$\partial_x \text{BesselJ}[0, x] = -\text{BesselJ}[1, x]$

$f[x] = \text{BesselJ}[1, x]$

$f[0] = 0$

and

$f'[x] = \partial_x \text{BesselJ}[1, x] =$

$f'[x] = \frac{1}{2} (\text{BesselJ}[0, x] - \text{BesselJ}[2, x])$

$f'[0] = \frac{1}{2}$