Mass-Spring Systems and Resonance

Comparing the effects of damping coefficients

An interesting problem is to compare the the effect of different values of the damping

coefficient c on the resulting motion of the mass on the spring. Consider the following problem: A 5 kg mass is attached to a spring with spring constant 8 newtons/meter. Determine the equation of motion which results if the initial displacement is y(0) = 6, the initial velocity is 0, there is no forcing, and the friction coefficient is

(a) ν = 0

(b) ν= 10

(c) ν = 15

In any case the differential equation is

$$5\, y''(t) + \nu y'(t) + 8\, y(t) = 0$$

$$y(0) = 6$$

$$y'(0) = 0$$

```
ClearAll["Global`*"];
Off[General::spell, General::spell1]
```

```
soleq1 = DSolve[ {5 y''[t] +0 y'[t] + 8 y[t] == 0,
                  y[0] == 6,
                  y'[0] == 0},
                  y[t],t]
```

$$\left\{\left\{y[t] \to 6 \,\mathrm{Cos}\left[2 \sqrt{\frac{2}{5}}\, t\right]\right\}\right\}$$

```
y1 = soleq1[[1,1,2]]
```

$$6 \,\mathrm{Cos}\left[2 \sqrt{\frac{2}{5}}\, t\right]$$

```
soleq2 = DSolve[ {5 y''[t] +10 y'[t] + 8 y[t] == 0,
                  y[0] == 6,
                  y'[0] == 0},
                  y[t],t]
```

$$\left\{\left\{y[t] \to 2\, e^{-t} \left(3 \,\mathrm{Cos}\left[\sqrt{\frac{3}{5}}\, t\right] + \sqrt{15}\, \mathrm{Sin}\left[\sqrt{\frac{3}{5}}\, t\right]\right)\right\}\right\}$$

```
y2 = soleq2[[1,1,2]]
```

$$2\, e^{-t} \left(3 \,\mathrm{Cos}\left[\sqrt{\frac{3}{5}}\, t\right] + \sqrt{15}\, \mathrm{Sin}\left[\sqrt{\frac{3}{5}}\, t\right]\right)$$

```
ComplexExpand[y2]
```

$$6\, e^{-t} \,\mathrm{Cos}\left[\sqrt{\frac{3}{5}}\, t\right] + 2 \sqrt{15}\, e^{-t} \,\mathrm{Sin}\left[\sqrt{\frac{3}{5}}\, t\right]$$

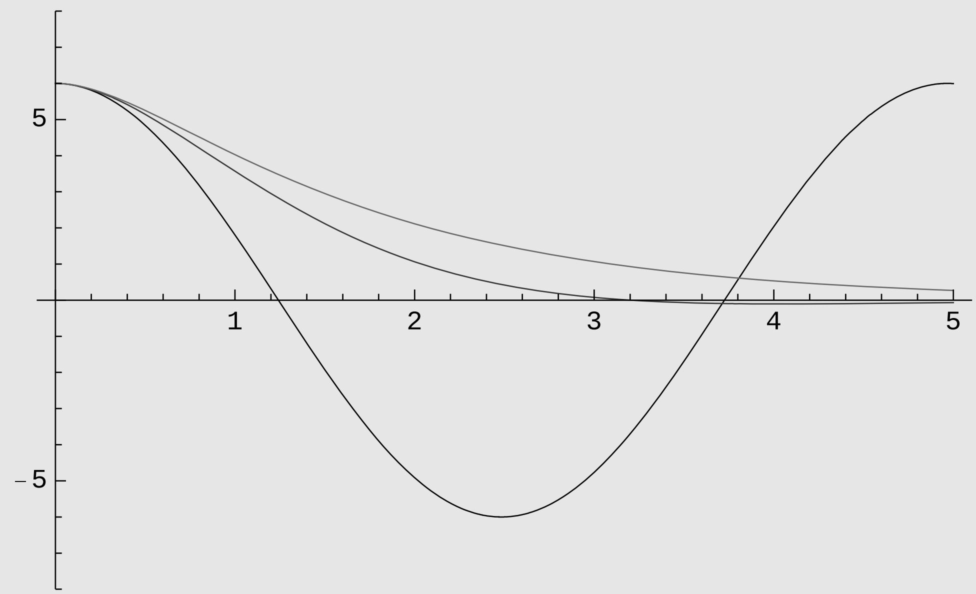
```
soleq3 = DSolve[ {5 y''[t] +15 y'[t] + 8 y[t] == 0,
               y[0] == 6,
               y'[0] == 0},
               y[t],t]
```

$$\left\{\left\{y[t] \to -\frac{3}{13}\left(-13\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t} + 3\sqrt{65}\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t} - 13\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t} - 3\sqrt{65}\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t}\right)\right\}\right\}$$

```
y3 = soleq3[[1,1,2]]
```

$$-\frac{3}{13}\left(-13\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t} + 3\sqrt{65}\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t} - 13\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t} - 3\sqrt{65}\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t}\right)$$

```
Plot[ {y1,y2,y3}, {t,0,5},
    PlotRange -> { -8,8},
        PlotStyle -> {GrayLevel[0], GrayLevel[.2],
                GrayLevel[.4], GrayLevel[.6]}]
```



Resonance as a result of damping and forcing

We consider a forced mass-spring oscillator system with friction or damping. The mass is taken to be 2 kg. The spring constant is taken to be 6 newtons/meter. The damping can be varied. The frequency of the forcing term can also be varied. We want to look at the response of the system to the this forcing under a variety of damping conditions.

The equation is

y''[t] + ν y'[t] + 6 y[t] = 2 Sin[a t]

y[0] = 0

y'[0] = 0

The forcing frequency coefficient is a.

The damping coefficient is ν.

```
Clear[y,t]
```

```
homeq = 2*y''[t] + ν y'[t] + 6 y[t] == 0
```

$$6\, y[t] + \nu\, y'[t] + 2\, y''[t] == 0$$

```
DSolve[homeq, y[t], t]
```

$$\left\{\left\{y[t] \to e^{\frac{1}{4} t \left(-\nu - \sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu + \sqrt{-48+\nu^2}\right)} C[2]\right\}\right\}$$

```
y0 = y[t] /. %
```

$$\left\{e^{\frac{1}{4} t \left(-\nu - \sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu + \sqrt{-48+\nu^2}\right)} C[2]\right\}$$

```
y0 = First[%]
```

$$e^{\frac{1}{4} t \left(-\nu - \sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu + \sqrt{-48+\nu^2}\right)} C[2]$$

```
yp = A Cos[a t] + B Sin[ a t]
```

$$A \operatorname{Cos}[a\, t] + B \operatorname{Sin}[a\, t]$$

```
D[yp, {t,2}] + ν D[yp, t] + 6 yp
```

$$-a^2 A \operatorname{Cos}[a\, t] - a^2 B \operatorname{Sin}[a\, t] + \nu\, (a B \operatorname{Cos}[a\, t] - a A \operatorname{Sin}[a\, t]) + 6\, (A \operatorname{Cos}[a\, t] + B \operatorname{Sin}[a\, t])$$

```
Collect[ %, {Cos[a t], Sin[a t]}]
```

$$\left(6\,A - a^2\,A + a\,B\,\nu\right)\,\text{Cos}[a\,t] + \left(6\,B - a^2\,B - a\,A\,\nu\right)\,\text{Sin}[a\,t]$$

```
uceqns = { 6 A - a^2 A + a B ν == 0, 6 B - a^2 B - a A ν == 2}
```

$$\left\{6\,A - a^2\,A + a\,B\,\nu == 0,\ 6\,B - a^2\,B - a\,A\,\nu == 2\right\}$$

```
coeffs = Solve[uceqns, {A, B}]
```

$$\left\{\left\{A \to -\frac{2\,a\,\nu}{36 - 12\,a^2 + a^4 + a^2\,\nu^2},\ B \to -\frac{2\,\left(-6 + a^2\right)}{36 - 12\,a^2 + a^4 + a^2\,\nu^2}\right\}\right\}$$

```
yp = First[ A Cos[ a t] + B Sin[ a t] /. coeffs]
```

$$-\frac{2\,a\,\nu\,\text{Cos}[a\,t]}{36 - 12\,a^2 + a^4 + a^2\,\nu^2} - \frac{2\,\left(-6 + a^2\right)\,\text{Sin}[a\,t]}{36 - 12\,a^2 + a^4 + a^2\,\nu^2}$$

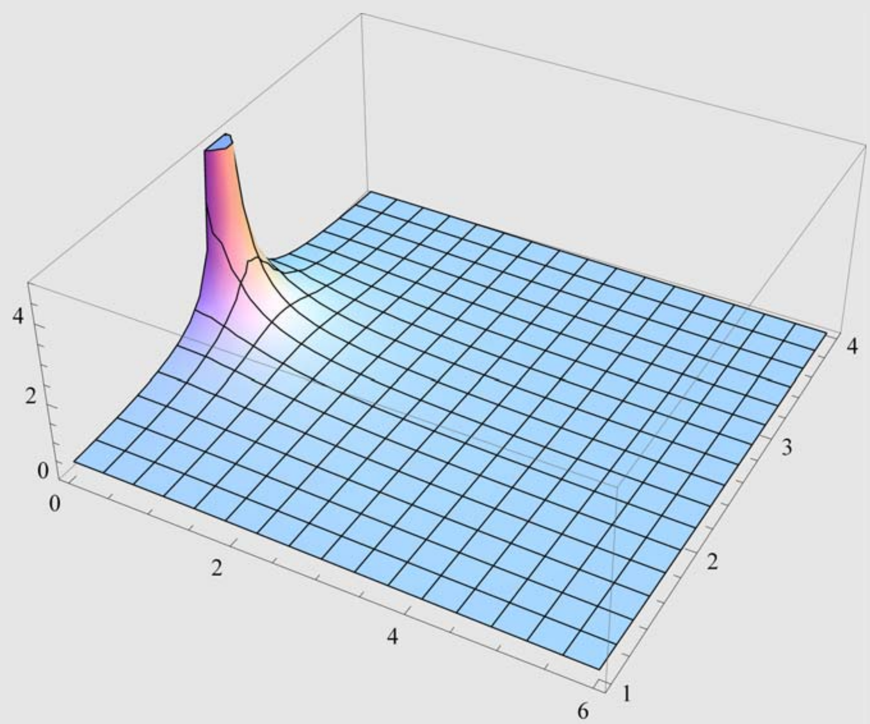
Having solved for the particular solution, I now put that particular solution in amplitude-phase form.

```
F0 = First[Sqrt[ A^2 + B^2] /. coeffs]//Simplify
```

$$2\,\sqrt{\frac{1}{36 + a^4 + a^2\,\left(-12 + \nu^2\right)}}$$

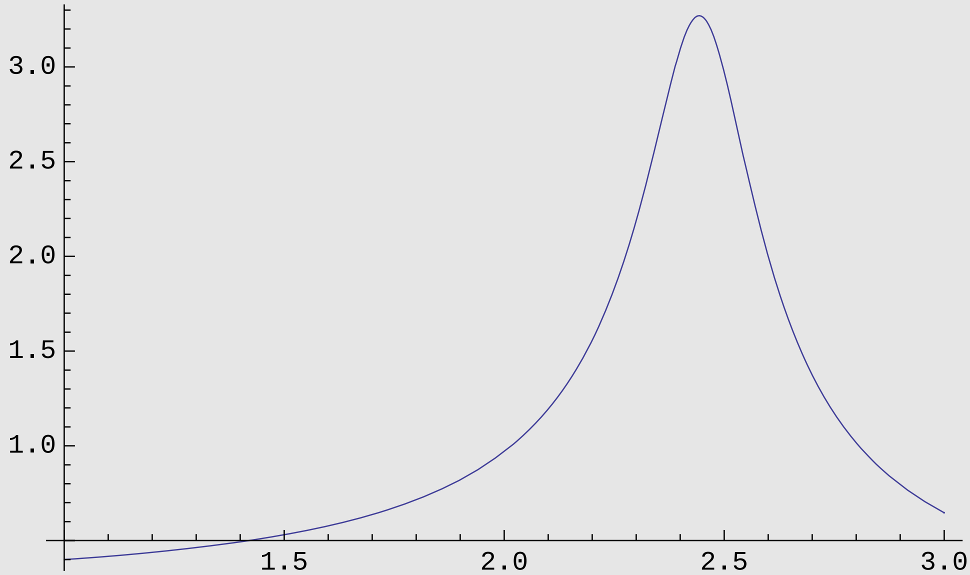
This then is the amplitude of the steady state response to the input forcing term. Now we wantt to plot the amplitude of the steady-state response as a function of the input frequency and the damping coefficient.

```
Plot3D[ F0, {ν, 0, 6}, {a, 1,4}, PlotRange -> {0, 5}]
```



Notice that something funny is happening near a = 2, and ν = 0.
The error messages suggest that the graph "hits" inifinity near here.
Notice too that the natural frequency of the undamped oscillator is 2!
This sounds suspiciously like the dictionary definition given above!
Lets plot the response as a function of forcing frequnecy for a damping coefficient near ν = 0.

```
Plot[F0 /. ν -> 0.25, {a, 1,3}]
```



The responce gets large near a = 2, and ν = 0.25. Let's actually find the full solution to the initial value problem.

```
y =(y0 + yp) /. {a -> 2, ν -> 0.25}
```

$e^{(-0.0625-1.73092\, i)\, t}\, C[1] + e^{(-0.0625+1.73092\, i)\, t}\, C[2] - 0.235294\, \text{Cos}[2\, t] + 0.941176\, \text{Sin}[2\, t]$

```
ComplexExpand[ y]
```

$e^{-0.0625\, t}\, C[1]\, \text{Cos}[1.73092\, t] + e^{-0.0625\, t}\, C[2]\, \text{Cos}[1.73092\, t] - 0.235294\, \text{Cos}[2\, t] + i \left(-e^{-0.0625\, t}\, C[1]\, \text{Sin}[1.73092\, t] + e^{-0.0625\, t}\, C[2]\, \text{Sin}[1.73092\, t]\right) + 0.941176\, \text{Sin}[2\, t]$

```
ic1 =( y /. t-> 0) == 0
```

$-0.235294 + C[1] + C[2] == 0$

```
ic2 = (D[y,t] /. t-> 0) == 0
```

$1.88235 - (0.0625 + 1.73092\,i)\,C[1] - (0.0625 - 1.73092\,i)\,C[2] == 0$

```
iceqns = { ic1, ic2}
```

$\{-0.235294 + C[1] + C[2] == 0,\ 1.88235 - (0.0625 + 1.73092\,i)\,C[1] - (0.0625 - 1.73092\,i)\,C[2] == 0\}$

```
Solve[ iceqns, {C[1], C[2]}]
```

$\{\{C[1] \to 0.117647 - 0.539495\,i,\ C[2] \to 0.117647 + 0.539495\,i\}\}$

```
y = y /. %
```

$\{(0.117647 - 0.539495\,i)\,e^{(-0.0625-1.73092\,i)\,t} + (0.117647 + 0.539495\,i)\,e^{(-0.0625+1.73092\,i)\,t} - 0.235294\,Cos[2\,t] + 0.941176\,Sin[2\,t]\}$

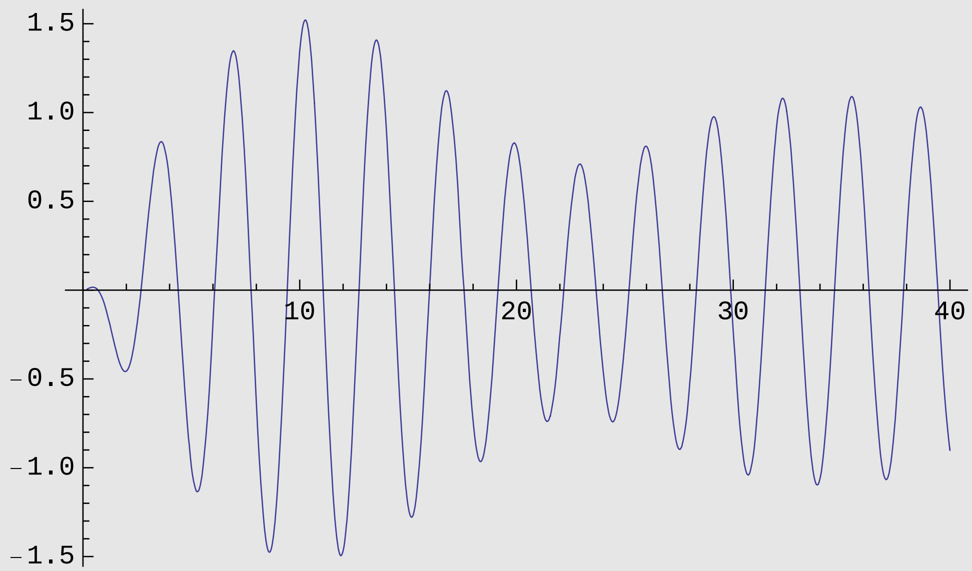
```
ComplexExpand[ First[%]]
```

$0.235294\,e^{-0.0625\,t}\,Cos[1.73092\,t] - 0.235294\,Cos[2\,t] - 1.07899\,e^{-0.0625\,t}\,Sin[1.73092\,t] + i\,(-1.11022\times10^{-16}\,e^{-0.0625\,t}\,Cos[1.73092\,t] + 1.38778\times10^{-17}\,e^{-0.0625\,t}\,Sin[1.73092\,t]) + 0.941176\,Sin[2\,t]$

```
trialsoln =Expand[%]
```

$(0.235294 - 1.11022\times10^{-16}\,i)\,e^{-0.0625\,t}\,Cos[1.73092\,t] - 0.235294\,Cos[2\,t] - (1.07899 - 1.38778\times10^{-17}\,i)\,e^{-0.0625\,t}\,Sin[1.73092\,t] + 0.941176\,Sin[2\,t]$

```
Plot[ trialsoln,{t, 0, 40},PlotPoints -> 50 ]
```



So the solution has been amplified ! The steady state amplitude is about double that of the input! Now let's see what happpens when there is no friction.

Forced, Undamped Motion

Let's see what happpens when there is no friction.

The equation has changed considerably, so we should properly re-solve from the beginning.

```
Clear[z,t]
```

```
hde = z''[t] + 6 z[t]  == 0
```

$6\,z[t] + z''[t] == 0$

```
DSolve[hde, z[t], t]
```

$\{\{z[t] \to C[1]\, Cos[\sqrt{6}\, t] + C[2]\, Sin[\sqrt{6}\, t]\}\}$

```
z0 = %[[1,1,2]]
```

$C[1]\, Cos[\sqrt{6}\, t] + C[2]\, Sin[\sqrt{6}\, t]$

```
zp = C t Cos[2 t] + D t Sin[2 t]
```

$C\, t\, Cos[2\, t] + D\, t\, Sin[2\, t]$

```
D[ zp, {t,2} ] + 6 zp
```

$4\, D\, Cos[2\, t] - 4\, C\, t\, Cos[2\, t] - 4\, C\, Sin[2\, t] - 4\, D\, t\, Sin[2\, t] + 6\, (C\, t\, Cos[2\, t] + D\, t\, Sin[2\, t])$

```
Simplify[%]
```

$2\, ((2\, D + C\, t)\, Cos[2\, t] + (-2\, C + D\, t)\, Sin[2\, t])$

```
zp = (-1/2) t Cos[ 2t]
```

$-\frac{1}{2}\, t\, Cos[2\, t]$

```
D[ zp, {t,2}] + 6 zp
```

$-t\, Cos[2\, t] + 2\, Sin[2\, t]$

```
z = z0 + zp
```

$-\frac{1}{2}\, t\, Cos[2\, t] + C[1]\, Cos[\sqrt{6}\, t] + C[2]\, Sin[\sqrt{6}\, t]$

```
ic1 = (z /. t -> 0) ==0
```

C[1] == 0

```
ic2 = (D[z,t] /. t -> 0) == 0
```

$-\frac{1}{2} + \sqrt{6}\, C[2] == 0$

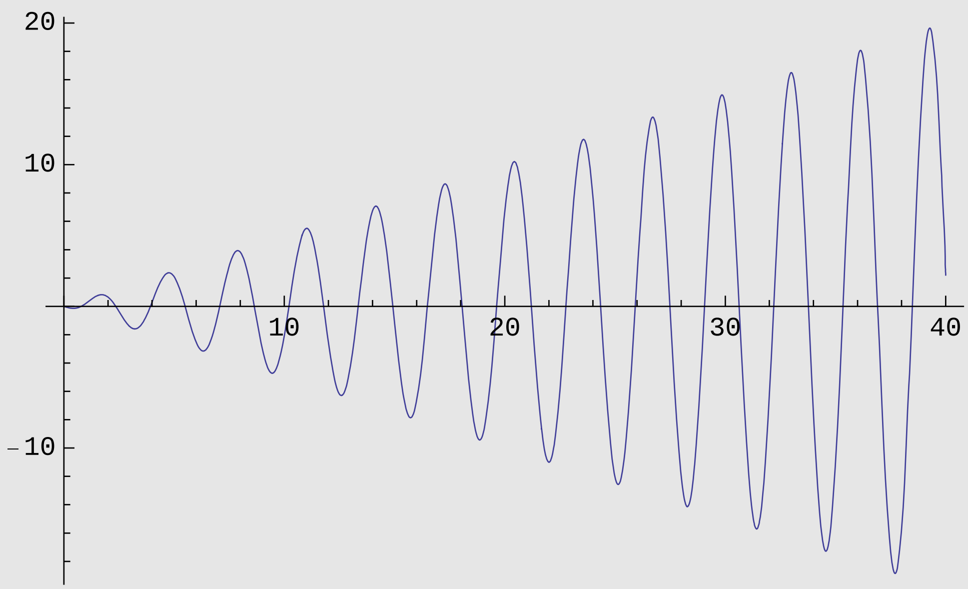
```
Solve[ {ic1, ic2}, {C[1], C[2]}]
```

$\left\{\left\{C[1] \to 0,\ C[2] \to \frac{1}{2\sqrt{6}}\right\}\right\}$

```
z = First[First[z /. %]]
```

$-\frac{1}{2}\, t\, \mathrm{Cos}[2\, t]$

```
Plot[ z, {t, 0, 40}, PlotPoints -> 50]
```



Notice that growing factor of t in the particualr solution. No wonder that the graphing complained about an infinite output response ampli tude when the friction coefficient b = 0.! This is the case of pure resonance, the previous graph was the case of damped resonance.

Exercise

Given the initial value problem

$$x''(t) + x(t) = 2\cos \omega t + 3\text{sim } \omega t$$

$$x(0) = 0$$

$$x'(0) = 0$$

Find a value of ω such that the solution has beats, and graph the solution.

Find a value such that the solution is resonance, and graph the solution.

Find a value of ω such that the solution is the superposition of two periodic functions, the ratio of whose periods is irrational. Graph the solution. Is the solution periodic?