Green's Theorem

Green's Theorem generalizes the case of an arbitrary smooth vector field. Instead of multiplying a constant curl by the area of the region, a varying (in general) curl is integrated over the region.

Green's Theorem. Let D be a plane region bounded by a sinple closed curve C having a piecewise smooth parameterization. Let $\vec{F}$ be a smooth vector field on D. Then

$$\int_D \operatorname{curl} \vec{F}(x, y)\, dA = \int_C \vec{F} \cdot d\vec{S}.$$

If we wrrite $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ and $\vec{S}(t) = (x(t), y(t))$. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C (P\, dx + Q\, dy).$$

Proof of Green's Theorem

Define

$$\vec{G}(x, y) = Q(x, y)\,\vec{i} - P(x, y)\,\vec{j}.$$

Then

$$\operatorname{curl} \vec{F} = \operatorname{div} \vec{G}.$$

According to the 2D Divergence Theorem,

$$\int_D \operatorname{div} \vec{G}(x, y)\, dA = \int_C \vec{G} \wedge d\vec{S}.$$

In terms of coordinates, the line integral on the right is

$$\int_a^b \left(Q\left(\vec{S}(t)\right) y'(t) - \left(-P\left(\vec{S}(t)\right)\right) x'(t)\right) dt \ =.$$

$$\int_a^b \left(P\left(\vec{S}(t)\right) x'(t) + Q\left(\vec{S}(t)\right) y'(t)\right) dt$$

The last integral is just the line integral $\int_C \vec{F} \cdot d\vec{S}$.

Curl of an affine vector field in the plane

Look at the total torque of a planar vector field

$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$

on a "jack" with horizontal and vertical arms centered at the origin.

Consider the special case that the components P and Q are both affine functions of (x, y).

```
P[x_, y_] := m11 x + m12 y + b1
Q[x_, y_] := m21 x + m22 y + b2
```

```
∫_{-h}^{h} Q[x, y] x ⅆx + ∫_{-h}^{h} - P[x, y] y ⅆy // Simplify
```

$$\frac{2}{3} h^3 (-m12 + m21)$$

Notice the values of:

```
{∂_x Q[x, y], ∂_y P[x, y]}
```

```
{m21, m12}
```

Thus the total torque is a multiple of:

```
∂x Q[x, y] - ∂y P[x, y]
```

```
-m12 + m21
```

The same factor, $m_{2\times 1} - m_{1\times 2}$ appears even if the center of the jack is at an arbitrary point (x_0, y_0):

```
∫_{-h}^{h} Q[x, y] (x - x0) ⅆx + ∫_{-h}^{h} - P[x, y] (y - y0) ⅆy
```

$$-\frac{2\,h^3\,m12}{3} + \frac{2\,h^3\,m21}{3} - 2\,b2\,h\,x0 - 2\,h\,m22\,x0\,y + 2\,b1\,h\,y0 + 2\,h\,m11\,x\,y0$$

Curl of an arbitrary vector field in the plane

Definition using cartesian coordinates

Generalize the preceding special case. For a 2-dimensional vector field

$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j},$$

its **curl** is the scalar field

$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P.$$

Here is the same thing in *Mathematica*:

```
curl[F_, coords_:{x, y}] := D[F[[2]], coords[[1]]] - D[F[[1]], coords[[2]]]
```

```
Clear[P, Q]
```

```
curl[{P[{x, y}], Q[{x, y}]}]
```

```
-P^({0,1})[{x, y}] + Q^({1,0})[{x, y}]
```

Using the familiar del operator

$$\nabla = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right)$$

and a symbolic determinant, the curl may be denoted by a symbolic wedge product:

$$\vec{F} = (P,\ Q) \implies \nabla \wedge \vec{F} = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right) \wedge (P,\ Q) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$$

Example

```
curl[{-y, x}]
```

```
2
```

Curl and total tangential force: affine vector field case

```
Clear[P, Q, R, S, x, y]
```

Consider again the special case that the components P and Q of the vector field $\vec{F}$ are both affine functions of $(x,\ y)$.

```
P[{x_, y_}] := P[x, y]
P[x_, y_] := m11 x + m12 y + b1
Q[{x_, y_}] := Q[x, y]
Q[x_, y_] := m21 x + m22 y + b2
F[{x_, y_}] := F[x, y]
F[x_, y_] := {P[{x, y}], Q[x, y]}
```

Consider a parameterized circle *C*:

```
x[t_] := R Cos[t]
y[t_] := R Sin[t]
S[t_] := {x[t], y[t]}
```

The line integral $\int_C \vec{F} \cdot d\vec{S}$:

$$\int_0^{2\pi} F[S[t]] . S'[t]\, dt$$

$(-m12 + m21)\,\pi\, R^2$

$$\text{curl}[F[x, y]]\,(\pi R^2)$$

$(-m12 + m21)\,\pi\, R^2$