The Burgers and KDV Equations

The Burgers Equation- Exact Solution

The solution of Burgers equation with a delta function as an initial condition is obtained most easilt by applying *Plot* to the expression in Equation

D[u[x, t], t] + u[x, t] D[u[x, t], x] -D[u[x, t], {x, 2}] == 0

We first check that this expression does in fact solve Burgers' equation:

In[88]:=
```
u[x_, t_] = (2 / ((1 / (E^(1 / 2) - 1)) +
   (1 / 2) Erfc[x / Sqrt[4 t]])) *
   (Exp[-x^2 / (4 t)] / Sqrt[4 Pi t])
```

Out[88]=

$$\frac{e^{-\frac{x^2}{4t}}}{\sqrt{\pi}\,\sqrt{t}\,\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\,\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}$$

In[89]:=
```
D[u[x, t], t]
```

Out[89]=

$$-\frac{e^{-\frac{x^2}{2t}}\,x}{4\,\pi\,t^2\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\,\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}-\frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\,\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}+\frac{e^{-\frac{x^2}{4t}}\,x^2}{4\sqrt{\pi}\,t^{5/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\,\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}$$

In[90]:=
```
u[x, t] D[u[x, t], x]
```

Out[90]=

$$\frac{e^{-\frac{x^2}{4t}}\left(\frac{e^{-\frac{x^2}{2t}}}{2\pi t\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}-\frac{e^{-\frac{x^2}{4t}}x}{2\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}\right)}{\sqrt{\pi}\sqrt{t}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}$$

In[91]:=
```
D[u[x, t], {x, 2}]
```

Out[91]=

$$\frac{e^{-\frac{x^2}{4t}}\left(\frac{e^{-\frac{x^2}{2t}}}{2\pi t\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^3}-\frac{e^{-\frac{x^2}{4t}}x}{4\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}\right)}{\sqrt{\pi}\sqrt{t}}-\frac{e^{-\frac{x^2}{2t}}x}{2\pi t^2\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}+\frac{-\frac{e^{-\frac{x^2}{4t}}}{2t}+\frac{e^{-\frac{x^2}{4t}}x^2}{4t^2}}{\sqrt{\pi}\sqrt{t}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}$$

In[92]:=
```
D[u[x, t], t] + u[x, t] D[u[x, t], x] -
  D[u[x, t], {x, 2}] == 0
```

Out[92]=

$$-\frac{e^{-\frac{x^2}{4t}}\left(\frac{e^{-\frac{x^2}{2t}}}{2\pi t\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^3}-\frac{e^{-\frac{x^2}{4t}}x}{4\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}\right)}{\sqrt{\pi}\sqrt{t}}+\frac{e^{-\frac{x^2}{2t}}x}{4\pi t^2\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}-\frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}+\frac{e^{-\frac{x^2}{4t}}x^2}{4\sqrt{\pi}\,t^{5/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}-\frac{-\frac{e^{-\frac{x^2}{4t}}}{2t}+\frac{e^{-\frac{x^2}{4t}}x^2}{4t^2}}{\sqrt{\pi}\sqrt{t}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}+\frac{e^{-\frac{x^2}{4t}}\left(\frac{e^{-\frac{x^2}{2t}}}{2\pi t\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)^2}-\frac{e^{-\frac{x^2}{4t}}x}{2\sqrt{\pi}\,t^{3/2}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}\right)}{\sqrt{\pi}\sqrt{t}\left(\frac{1}{-1+\sqrt{e}}+\frac{1}{2}\text{Erfc}\left[\frac{x}{2\sqrt{t}}\right]\right)}==0$$

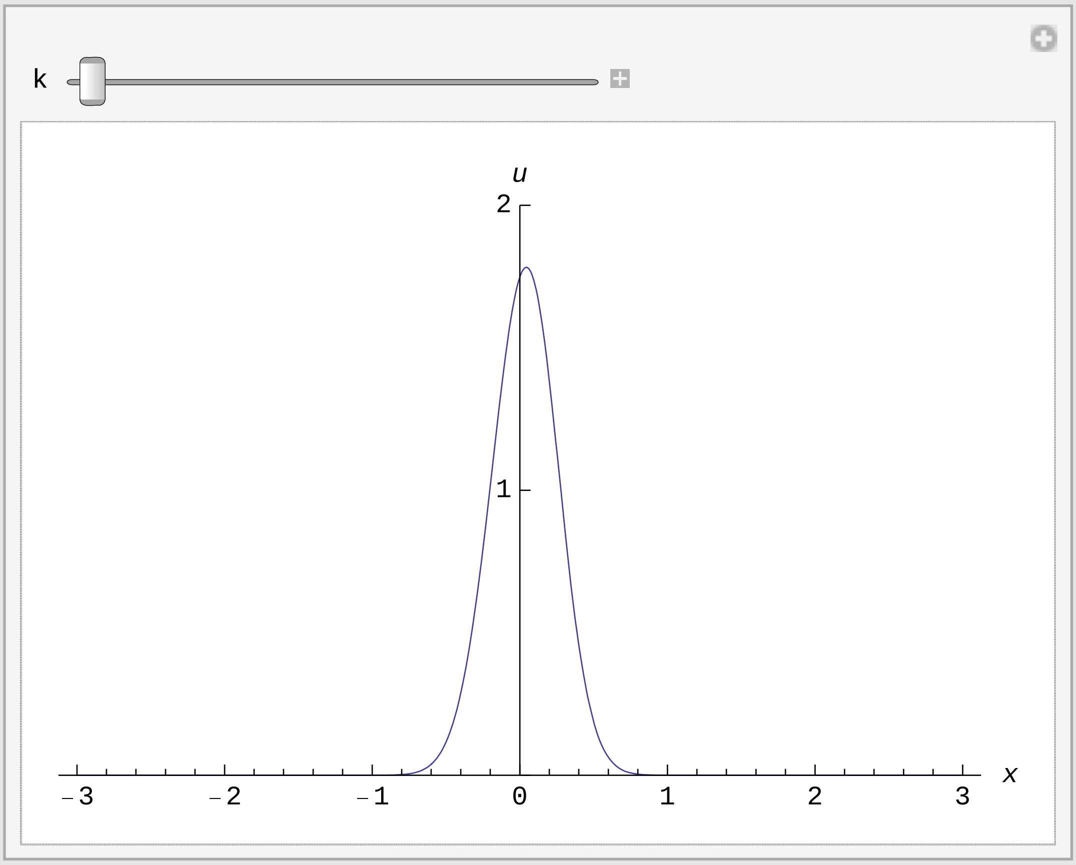
In[93]:=
```
Together[D[u[x, t], t] + u[x, t] D[u[x, t], x] -
   D[u[x, t], {x, 2}]] /. Sqrt[Pi t] -> Sqrt[Pi] Sqrt[t]
```

Out[93]=
```
0
```

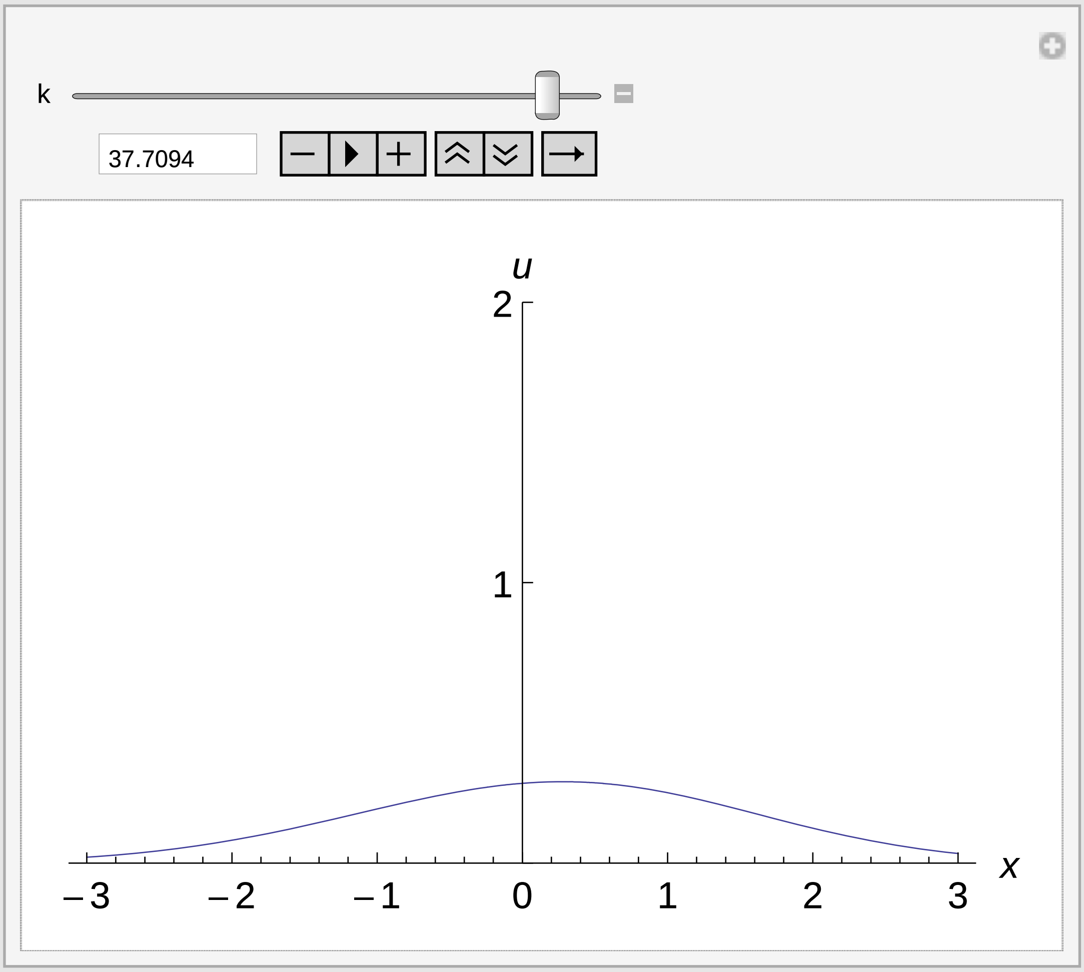
This solution is displayed for various times with the commands shown below. Once generated, the sequence can be animated.

In[94]:=
```
Manipulate[Plot[u[x, 0.025 * k], {x, -3, 3},
    PlotRange → {0, 2},
    Ticks → {Automatic, Range[0, 2, 1]},
    AxesLabel → {x, u}], {k, 1, 40}]
```

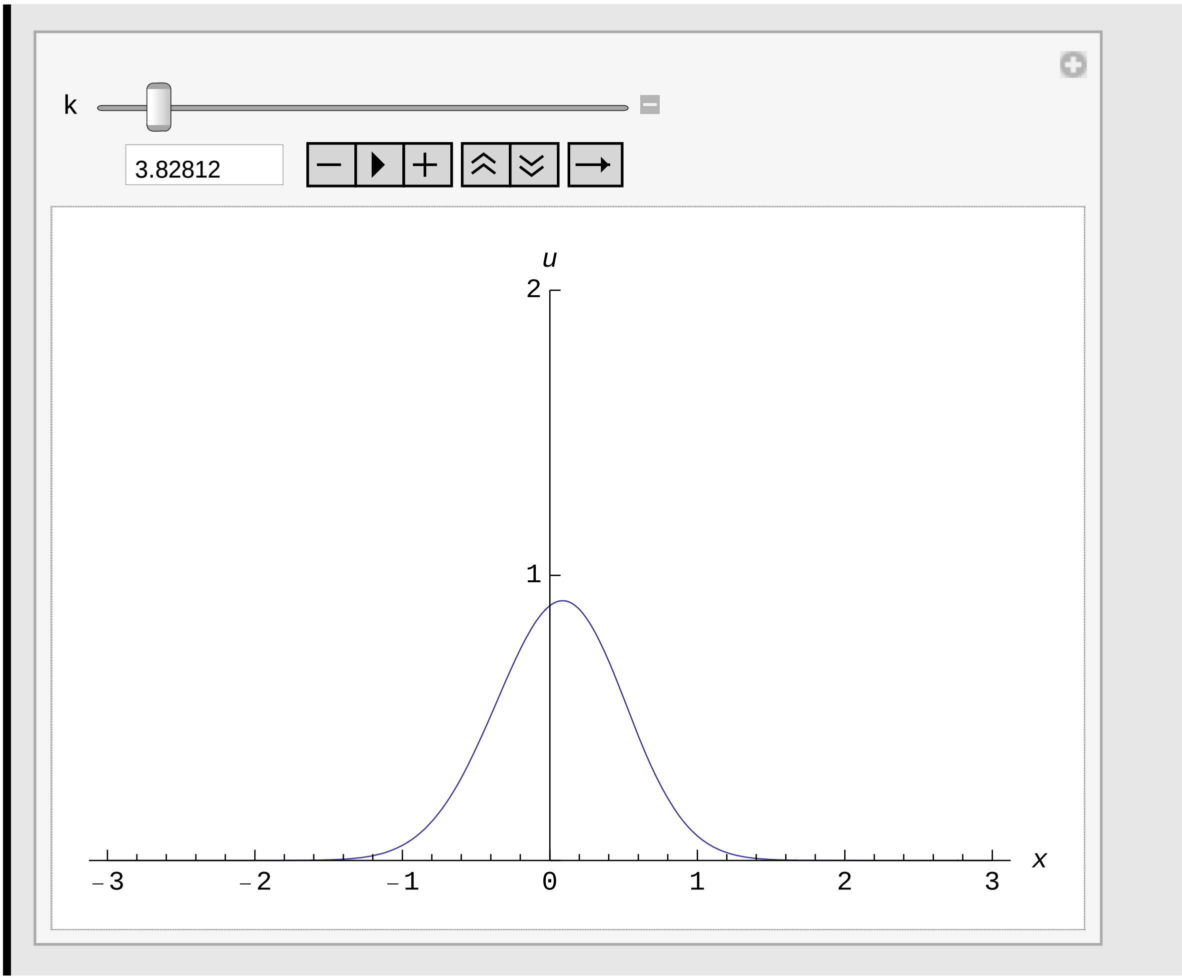
Out[94]=



In[55]:=



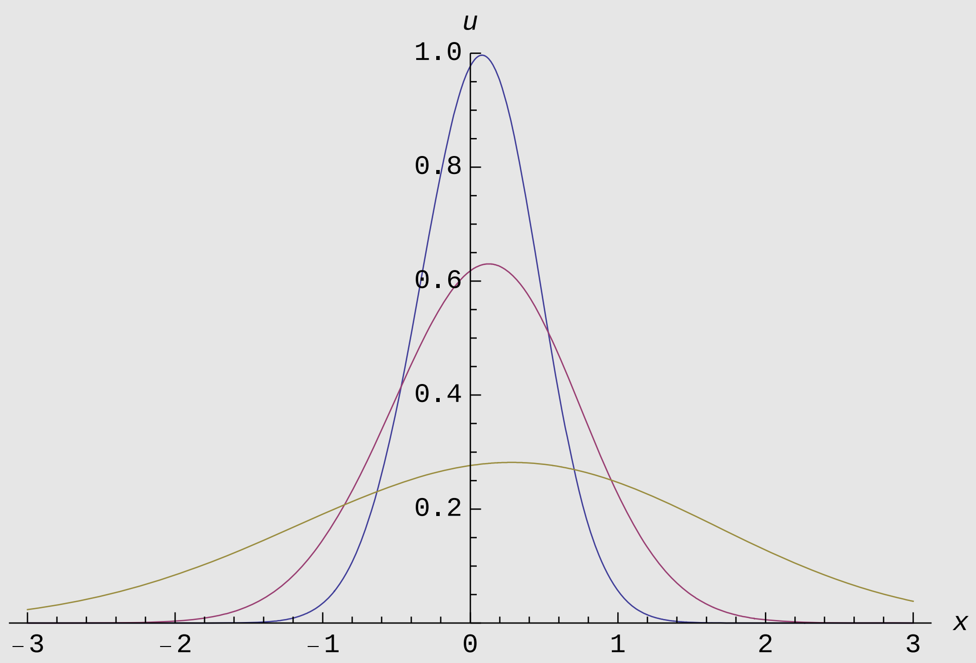
Out[55]=



This diagram is generated by plotting the function *u[x,t]* for the indicated times:

In[56]:=
```
Plot[{u[x, 0.08], u[x, 0.2], u[x, 1]}, {x, -3, 3},
  PlotRange -> {0, 1}, AxesLabel -> {x, u}]
```

Out[56]=



Elementary Soliton Solutions

The two panels of this figure are generated by first constructing the soliton and anti-soliton solutions in (9.54) and (9.55) and then using *Plot* to produce the line plots. The velocity has been set equal to *1/2*.

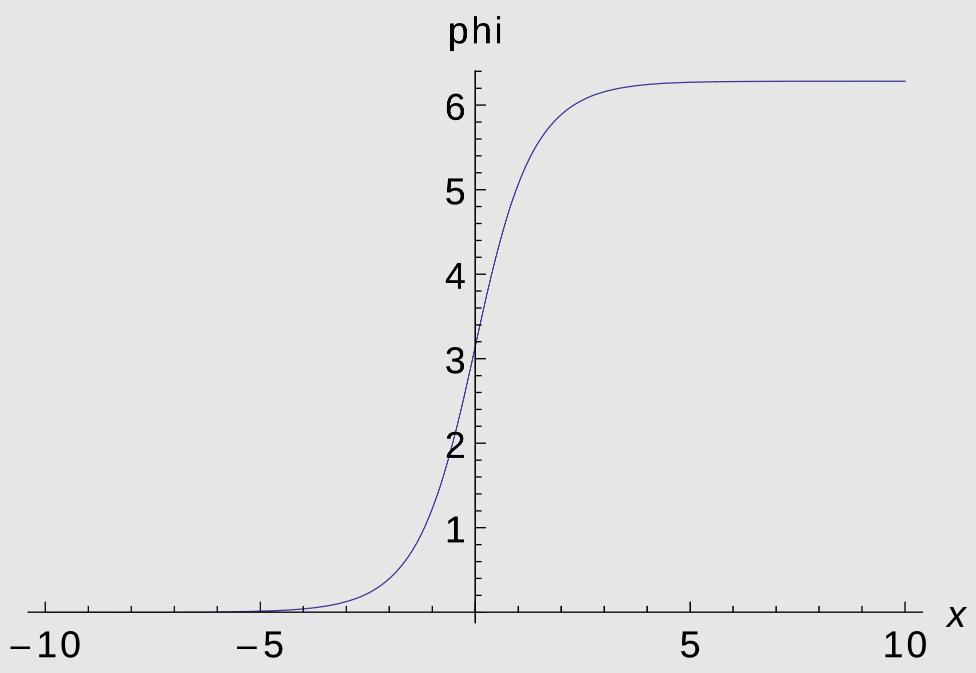
In[57]:=
```
phi1[x_,t_,v_]:=4ArcTan[Exp[(x-v t)/Sqrt[1-v^2]]]
```

In[58]:=
```
phi2[x_,t_,v_]:=4ArcCot[Exp[(x-v t)/Sqrt[1-v^2]]]
```

In[59]:=

```
Plot[phi1[x,0,1/2],{x,-10,10},AxesLabel->{x,phi}]
```

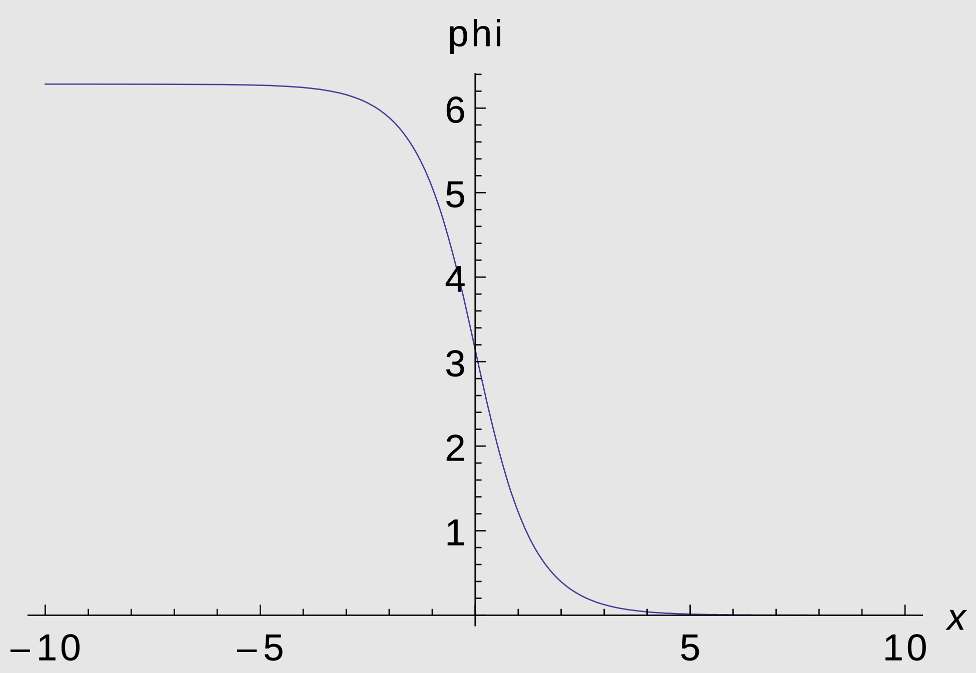
Out[59]=



In[60]:=

```
Plot[phi2[x,0,1/2],{x,-10,10},AxesLabel->{x,phi}]
```

Out[60]=



Traveling wave solution for KDV-Burgers equation

The form of compound KdV-Burgers equation involving nonlinear and dissipation effects is:

D[u[x, t], t] + αu[x, t] D[u[x, t], x] +βu[x, t]^2 D[u[x, t], x]+γ- D[u[x, t], {x, 2}] -θD[u[x, t], {x, 3}]== 0

Solving the above set of equations by using the Wu elimination method we can obtain the following solutions

In[61]:=
```
Clear["Global`*"];
```

In[62]:=
```
ξ = λ * (x - k * t + C0)
```

Out[62]=
```
λ (-k t + x + C0)
```

In[63]:=
```
φ[ξ_] := Sum[Sin[ω[ξ]]^(i-1) * (Bi * Sin[ω[ξ]] + Ai * Cos[ω[ξ]]) + A0, {i, 1}]
```

In[64]:=
```
φ[ξ_]
```

Out[64]=
```
A0 + Cos[ω[ξ_]] A1 + Sin[ω[ξ_]] B1
```

In[65]:=
```
u[x_, t_] := φ[ξ]
```

In[66]:=
```
φ[ξ]
```

Out[66]=
```
A0 + Cos[ω[λ (-k t + x + C0)]] A1 + Sin[ω[λ (-k t + x + C0)]] B1
```

In[67]:=
```
ω'[ξ] := Cos[ω[ξ]]
```

In[68]:=
```
t[ω[ξ]] = Dt[φ[ξ], t, Constants → {A0, A1, B1, λ, k, C0}] /.
  {ω'[ξ] → Cos[ω[ξ]], Dt[x, t, Constants → {A0, A1, B1, λ, k, C0}] → 0}
```

Out[68]=
```
k λ Cos[ω[λ (-k t + x + C0)]] Sin[ω[λ (-k t + x + C0)]] A1 -
  k λ Cos[ω[λ (-k t + x + C0)]]^2 B1
```

In[69]:=
```
x1[ω[ξ]] = Dt[φ[ξ], x, Constants → {A0, A1, B1, λ, k, C0}] /.
  {ω'[ξ] → Cos[ω[ξ]], Dt[t, x, Constants → {A0, A1, B1, λ, k, C0}] → 0}
```

Out[69]=
```
-λ Cos[ω[λ (-k t + x + C0)]] Sin[ω[λ (-k t + x + C0)]] A1 +
 λ Cos[ω[λ (-k t + x + C0)]]^2 B1
```

In[70]:=
```
x2[ω[ξ]] =
 Collect[
   Dt[φ[ξ], {x, 2}, Constants → {A0, A1, B1, λ, k, C0}] /.
       {ω'[ξ] → Cos[ω[ξ]],
        Dt[t, {x, 2}, Constants → {A0, A1, B1, λ, k, C0}] → 0} /.
      {ω'[ξ] → Cos[ω[ξ]], Dt[t, x, Constants → {A0, A1, B1, λ, k, C0}] →
        0} /. {ω'[ξ] → Cos[ω[ξ]],
      Dt[φ[ξ], {x, 2}, Constants → {A0, A1, B1, λ, k, C0}] → 0} /.
    Sin[ω[ξ]]^3 → Sin[ω[ξ]] - Sin[ω[ξ]] * Cos[ω[ξ]]^2,
   {Sin[ω[ξ]]^(_ : 1) Cos[ω[ξ]]^(_ : 1),
    Cos[ω[ξ]]^(_ : 1) Sin[ω[ξ]]^(_ : 1)}] /. Sin[ω[ξ]]^2 → 1 - Cos[ω[ξ]]^2
```

Out[70]=
```
-λ^2 Cos[ω[λ (-k t + x + C0)]]^3 A1 +
 λ^2 Cos[ω[λ (-k t + x + C0)]] (1 - Cos[ω[λ (-k t + x + C0)]]^2) A1 -
 2 λ^2 Cos[ω[λ (-k t + x + C0)]]^2 Sin[ω[λ (-k t + x + C0)]] B1
```

In[71]:=
```
x3[ω[ξ]] =
 Collect[
   Dt[φ[ξ], {x, 3}, Constants → {A0, A1, B1, λ, k, C0}] /.
        {ω'[ξ] → Cos[ω[ξ]],
         Dt[t, {x, 3}, Constants → {A0, A1, B1, λ, k, C0}] → 0} /.
       {ω'[ξ] → Cos[ω[ξ]],
        Dt[t, {x, 2}, Constants → {A0, A1, B1, λ, k, C0}] → 0} /.
      {ω'[ξ] → Cos[ω[ξ]], Dt[t, x, Constants → {A0, A1, B1, λ, k, C0}] →
        0} /. {ω'[ξ] → Cos[ω[ξ]],
      Dt[φ[ξ], {x, 2}, Constants → {A0, A1, B1, λ, k, C0}] → 0} /.
    Sin[ω[ξ]]^3 → Sin[ω[ξ]] - Sin[ω[ξ]] * Cos[ω[ξ]]^2,
   {Sin[ω[ξ]]^(_ : 1) Cos[ω[ξ]]^(_ : 1),
    Cos[ω[ξ]]^(_ : 1) Sin[ω[ξ]]^(_ : 1)}] /. Sin[ω[ξ]]^2 → 1 - Cos[ω[ξ]]^2
```

Out[71]=
```
-λ^3 Cos[ω[λ (-k t + x + C0)]] Sin[ω[λ (-k t + x + C0)]] A1 +
 6 λ^3 Cos[ω[λ (-k t + x + C0)]]^3 Sin[ω[λ (-k t + x + C0)]] A1 -
 2 λ^3 Cos[ω[λ (-k t + x + C0)]]^4 B1 +
 4 λ^3 Cos[ω[λ (-k t + x + C0)]]^2 (1 - Cos[ω[λ (-k t + x + C0)]]^2) B1
```

In[72]:=

```
eq[ω[ξ]] = t[ω[ξ]] + p * φ[ξ] * x1[ω[ξ]] + q * (φ[ξ])^2 * x1[ω[ξ]] +
   r * x2[ω[ξ]] - s * x3[ω[ξ]]
```

Out[72]=

$$\begin{aligned}
&k\,\lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1-\\
&\;k\,\lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,B_1-\\
&\;s\,\Big(-\lambda^3\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\\
&\quad 6\,\lambda^3\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^3\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1-\\
&\quad 2\,\lambda^3\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^4\,B_1+\\
&\quad 4\,\lambda^3\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,\big(1-\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\big)\,B_1\Big)+\\
&\;p\,\Big(-\lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\\
&\quad \lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,B_1\Big)\\
&\;\;(A_0+\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,B_1)+\\
&\;q\,\Big(-\lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\\
&\quad \lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,B_1\Big)\\
&\;\;(A_0+\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,B_1)^2+\\
&\;r\,\Big(-\lambda^2\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^3\,A_1+\\
&\quad \lambda^2\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\big(1-\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\big)\,A_1-\\
&\quad 2\,\lambda^2\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,B_1\Big)
\end{aligned}$$

In[73]:=

```
vect =
 Collect[eq[ω[ξ]], {Sin[ω[ξ]]^(_ : 1) * Cos[ω[ξ]]^(_ : 1),
    Sin[ω[ξ]]^(_ : 1), Cos[ω[ξ]]^(_ : 1)}] /.
  Sin[ω[ξ]]^2 → 1 - Cos[ω[ξ]]^2
```

Out[73]=

$$\begin{aligned}
&r\,\lambda^2\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,A_1+\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\\
&\quad \mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\big(k\,\lambda\,A_1+s\,\lambda^3\,A_1-p\,\lambda\,A_0\,A_1-q\,\lambda\,A_0^2\,A_1\big)-\\
&\;q\,\lambda\,\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]^3\,A_1\,B_1^2+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,\big(-k\,\lambda\,B_1-4\,s\,\lambda^3\,B_1+p\,\lambda\,A_0\,B_1+q\,\lambda\,A_0^2\,B_1\big)+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]\\
&\quad \big(1-\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\big)\,(-p\,\lambda\,A_1\,B_1-2\,q\,\lambda\,A_0\,A_1\,B_1)+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^3\,\big(-2\,r\,\lambda^2\,A_1+p\,\lambda\,A_1\,B_1+2\,q\,\lambda\,A_0\,A_1\,B_1\big)+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^4\,\big(6\,s\,\lambda^3\,B_1+q\,\lambda\,A_1^2\,B_1\big)+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,\mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\\
&\quad \big(-p\,\lambda\,A_1^2-2\,q\,\lambda\,A_0\,A_1^2-2\,r\,\lambda^2\,B_1+p\,\lambda\,B_1^2+2\,q\,\lambda\,A_0\,B_1^2\big)+\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^3\\
&\quad \mathrm{Sin}[\omega[\lambda\,(-k\,t+x+C_0)]]\,\big(-6\,s\,\lambda^3\,A_1-q\,\lambda\,A_1^3+2\,q\,\lambda\,A_1\,B_1^2\big)+\\
&\;\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\,\big(1-\mathrm{Cos}[\omega[\lambda\,(-k\,t+x+C_0)]]^2\big)\,\big(-2\,q\,\lambda\,A_1^2\,B_1+q\,\lambda\,B_1^3\big)
\end{aligned}$$

In[74]:=
```
vect1 =
 Collect[vect /. {Sin[ω[ξ]]^3 → Sin[ω[ξ]] - Sin[ω[ξ]] * Cos[ω[ξ]]^2},
  {Sin[ω[ξ]]^(_ : 1) * Cos[ω[ξ]]^(_ : 1), Sin[ω[ξ]]^(_ : 1),
   Cos[ω[ξ]]^(_ : 1)}]
```

Out[74]=

$$\text{Cos}[\omega[\lambda(-k t + x + C_0)]] \left(r \lambda^2 A_1 - p \lambda A_1 B_1 - 2 q \lambda A_0 A_1 B_1\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]]^3 \left(-2 r \lambda^2 A_1 + 2 p \lambda A_1 B_1 + 4 q \lambda A_0 A_1 B_1\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]]^2 \,\text{Sin}[\omega[\lambda(-k t + x + C_0)]] \left(-p \lambda A_1^2 - 2 q \lambda A_0 A_1^2 - 2 r \lambda^2 B_1 + p \lambda B_1^2 + 2 q \lambda A_0 B_1^2\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]] \,\text{Sin}[\omega[\lambda(-k t + x + C_0)]] \left(k \lambda A_1 + s \lambda^3 A_1 - p \lambda A_0 A_1 - q \lambda A_0^2 A_1 - q \lambda A_1 B_1^2\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]]^3 \,\text{Sin}[\omega[\lambda(-k t + x + C_0)]] \left(-6 s \lambda^3 A_1 - q \lambda A_1^3 + 3 q \lambda A_1 B_1^2\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]]^4 \left(6 s \lambda^3 B_1 + 3 q \lambda A_1^2 B_1 - q \lambda B_1^3\right) + \text{Cos}[\omega[\lambda(-k t + x + C_0)]]^2 \left(-k \lambda B_1 - 4 s \lambda^3 B_1 + p \lambda A_0 B_1 + q \lambda A_0^2 B_1 - 2 q \lambda A_1^2 B_1 + q \lambda B_1^3\right)$$

In[75]:=
```
s1 = Collect[(r λ^2 A1 - p λ A1 B1 - 2 q λ A0 A1 B1) / (λ * A1) // Simplify, {p, q}]
```

Out[75]= $r \lambda - p B_1 - 2 q A_0 B_1$

In[76]:=
```
s2 = Collect[(-2 r λ^2 A1 + 2 p λ A1 B1 + 4 q λ A0 A1 B1) / (λ * A1) // Simplify, q]
```

Out[76]= $-2 r \lambda + 2 p B_1 + 4 q A_0 B_1$

In[77]:=
```
s3 = Collect[(-p λ A1^2 - 2 q λ A0 A1^2 - 2 r λ^2 B1 + p λ B1^2 + 2 q λ A0 B1^2) / (λ) //
   Simplify, {q, p}]
```

Out[77]= $-2 r \lambda B_1 + p \left(-A_1^2 + B_1^2\right) + q \left(-2 A_0 A_1^2 + 2 A_0 B_1^2\right)$

In[78]:=
```
s4 = Collect[(k λ A1 + s λ^3 A1 - p λ A0 A1 - q λ A0^2 A1 - q λ A1 B1^2) / (λ) // Simplify,
  {q, p}]
```

Out[78]= $\left(k + s \lambda^2\right) A_1 - p A_0 A_1 + q A_1 \left(-A_0^2 - B_1^2\right)$

In[79]:=
```
s5 = Collect[(-6 s λ^3 A1 - q λ A1^3 + 3 q λ A1 B1^2) / (λ * A1) // Simplify, q]
```

Out[79]= $-6 s \lambda^2 + q \left(-A_1^2 + 3 B_1^2\right)$

In[80]:=
```
s6 = (6 s λ^3 B1 + 3 q λ A1^2 B1 - q λ B1^3) / (λ * B1) // Simplify
```

Out[80]= $6 s \lambda^2 + 3 q A_1^2 - q B_1^2$

In[81]:=
```
s7 = (-k λ B1 - 4 s λ^3 B1 + p λ A0 B1 + q λ A0^2 B1 - 2 q λ A1^2 B1 + q λ B1^3) / (λ * B1) //
  Simplify
```

Out[81]= $-k - 4 s \lambda^2 + p A_0 + q A_0^2 - 2 q A_1^2 + q B_1^2$

In[82]:=
```
vect = {s1, s2, s3, s4, s5, s6, s7} /. {A1 → 0}
```

Out[82]= $\{r \lambda - p B_1 - 2 q A_0 B_1, -2 r \lambda + 2 p B_1 + 4 q A_0 B_1, -2 r \lambda B_1 + p B_1^2 + 2 q A_0 B_1^2, 0, -6 s \lambda^2 + 3 q B_1^2, 6 s \lambda^2 - q B_1^2, -k - 4 s \lambda^2 + p A_0 + q A_0^2 + q B_1^2\}$

In[83]:=
```
sol1 = Solve[vect[[6]] == 0, B1]
```

Out[83]= $\left\{\left\{B_1 \to -\frac{\sqrt{6}\sqrt{s}\,\lambda}{\sqrt{q}}\right\}, \left\{B_1 \to \frac{\sqrt{6}\sqrt{s}\,\lambda}{\sqrt{q}}\right\}\right\}$

In[84]:=
```
vect1 = vect[[1]]
```

Out[84]= $r \lambda - p B_1 - 2 q A_0 B_1$

In[85]:=
```
sol2 = Collect[Solve[vect1 == 0, A0] /. B1 → (√6 √s λ)/√q, {p, q}]
```

Out[85]= $\left\{\left\{A_0 \to -\frac{p}{2 q} + \frac{r}{2\sqrt{6}\sqrt{q}\sqrt{s}}\right\}\right\}$

In[86]:=
```
vect2 = vect[[7]] /. {B1 → (√6 √s λ)/√q, A0 → -(-r λ + (√6 p √s λ)/√q)/(2 √6 √q √s λ)} // Simplify
```

Out[86]= $-k - \frac{p^2}{4 q} + \frac{r^2}{24 s} + 2 s \lambda^2$

In[87]:=
```
Solve[vect2 == 0, k] // Simplify
```

Out[87]=
$$\left\{\left\{k \to -\frac{p^2}{4\,q} + \frac{r^2}{24\,s} + 2\,s\,\lambda^2\right\}\right\}$$